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FOUNDATIONS

*Scholz, Heinrich. **Vorlesungen über Grundzüge der mathematischen Logik. Teil I.** 2d ed. Ausarbeitungen mathematischer und physikalischer Vorlesungen. Band VI. Aschendorffsche Verlagsbuchhandlung, Münster, 1950. xiv+276 pp. 10 DM.

This is a treatise on the classical (two-valued) propositional algebra and the predicate calculus of first order based on it. The author's point of view may be described as semantical. He presupposes "eine präzisierte Metasprache für welche die zu formalisierende Logik verbindlich ist." As he points out in the preface, this implies a certain metaphysical bias. The fundamental idea is that of semantical "Allgemeingültigheit", i.e. validity for every possible model. The author develops both the algebra and the calculus from this point of view; the deductive, axiomatic standpoint is not considered at all for the algebra, and is relegated to a secondary role for the calculus. For the algebra this amounts to a treatment from the matrix point of view; but for the calculus it entails that the line between constructive and nonconstructive results is not as sharply drawn as one would expect in a modern work. Furthermore the theory is tied to a specific interpretation; there is no consideration of generalized (intuitionistic etc.) systems, or of relations to algebraic systems (lattices etc.). Within these limitations the treatment is very full and detailed. The author introduces an elaborate notation into the language being used. This makes for great preciseness in some places, but entails excessive prolixity in others. The treatment of the predicate calculus includes the principal theorems of Löwenheim and Skolem, the completeness theorem of Gödel, and a citation, without proof, of the nondecidability theorem of Church.

H. B. Curry (State College, Pa.).

Kalmár, László, and Surányi, János. On the reduction of the decision problem. III. Pepis prefix, a single binary predicate. J. Symbolic Logic 15, 161–173 (1950).

In previous papers with the same main title [L. Kalmár, same J. 4, 1–9 (1939); L. Kalmár and J. Surányi, ibid. 12, 65–73 (1947); these Rev. 11, 303] it has been shown that every formula of the first-order predicate calculus is equivalent to one containing no predicate variable but a single binary one, and with prefix of the form $(Ex_1)(x_2)(Ex_3)(x_4)\cdots(x_n)$, called the Ackermann prefix, likewise with prefix $(x_1)(x_2)(x_3)(Ex_4)\cdots(Ex_n)$, called the Gödel prefix. In the present paper the corresponding result is proved for prefixes of the form $(x_1)(x_2)(Ex_3)(x_4)\cdots(x_n)$, called the Pepis prefix. This prefix contains only a single existential quantifier. *O. Frink* (State College, Pa.).

Kalmár, László. Contributions to the reduction theory of the decision problem. I. Prefix $(x_1)(x_2)(Ex_3)\cdots(Ex_{n-1})(x_n)$, a single binary predicate. Acta Math. Acad. Sci. Hungar. 1, 64–73 (1950). (English. Russian summary)

Continuing his work of finding simple standard reduction types [see the preceding review], the author shows that every formula of the first-order predicate calculus is equivalent to a formula containing no predicate variable except a single binary one, and with prefix of the form $(x_1)(x_2)(Ex_3)\cdots(Ex_{n-1})(x_n)$. *O. Frink.*

Pil'čak, B. Yu. On the decision problem for the calculus of problems. Doklady Akad. Nauk SSSR (N.S.) 75, 773–776 (1950). (Russian)

An interesting direct decision method for the intuitionistic calculus of propositions, which Kolmogoroff interpreted as a calculus of problems. *A. Heyting* (Amsterdam).

Markov, A. The impossibility of certain algorithms in the theory of associative systems. Doklady Akad. Nauk SSSR (N.S.) 77, 19–20 (1951). (Russian)

In this review (but not in the paper) $[C, A]$ or $[C]$ denotes the associative system determined in the alphabet A by the set of relations C. The problem of the existence of algorithms for deciding the following questions is considered: given an alphabet A and finite sets, C_1 and C_2, of relations, (1) are $[C_1]$ and $[C_2]$ isomorphic? (2) does $[C_1]$ contain only one member? (3) is it a group? (4) contained in a group? (5) is it a semi-group [with cancellation]? (6) ("the meta-problem of identity") is $[C_1]$ a system in which the identity-problem [=word-problem] is solvable? A negative answer is given in all six cases: no such algorithms exist for an alphabet with 4 or more letters.

An outline of the proof is given. A system $[C_k, A_k]$ of an earlier paper [same Doklady (N.S.) 58, 353–356 (1947); these Rev. 9, 321] is first transformed by Post's method [E. L. Post, Bull. Amer. Math. Soc. 50, 284–316 (1944); these Rev. 6, 29] into a system $[D, A_0]$ where A_0 is the 2-letter alphabet $\{a, b\}$. Given any two words G, H of A_0, let $D_{G,H}$ be the set of relations in $B=\{a, b, c, d\}$ formed by adjoining the relations $cGd \leftrightarrow 0$ and $\xi cHd \leftrightarrow cHd$ ($\xi = a, b, c, d$) to D. In the system $[D_{G,H}, B]$ the answer to questions (2) to (6) is "Yes" if, and only if, $G \leftrightarrow H$ in $[D, A_0]$. Since $[C_k, A_k]$, and therefore also $[D, A_0]$, has an unsolvable word problem [Markov, as above] this proves the unsolvability of (2) to (6), and that of (1) follows.

M. H. A. Newman (Manchester).

Rasiowa, H., and Sikorski, R. A proof of the completeness theorem of Gödel. Fund. Math. 37, 193–200 (1950).

Following Lindenbaum, let $E(\beta)$ denote the class of all formulas γ which are equivalent to β, where β and γ are formulas of the first order functional calculus. If we define $E(\beta) \cup E(\gamma) = E(\beta \vee \gamma)$ and $E(\beta)' = E(\sim\beta)$, then the set of all classes $E(\beta)$ forms a Boolean algebra B^*. Using Stone's theorem and the category method of argument it is shown that for any nonprovable formula α there is a prime ideal p^* in B^* such that in the 2-element Boolean algebra B^*/p^* we have $[E(\alpha)]=0$ and for all β, $\bigcup_{r \in I}[E(\beta(\frac{x_r}{x_k}))]=[E(\forall_{x_k}\beta)]$, where I = the set of integers, $\beta(\frac{x_r}{x_k})$ is any formula obtained from β by substituting x_r for all occurrences of x_k in any alphabetic variant of β in which x_r is not bound, and

$[E(\beta)]$ is the element of B^*/p^* determined by $E(\beta)$. By methods of Mostowski we can define a function $\Phi_\beta{}^0$ of formulas β onto the 2-element Boolean algebra with elements 0 and 1 such that $\Phi_\beta{}^0$ assumes the value 1 (0) if and only if β ($\sim\beta$) is satisfiable. Then it is shown that in B^*/p^* we have $\Phi_\beta{}^0=[E(\beta)]$ for all β. In particular, $\Phi_\alpha{}^0=[E(\alpha)]=0$, i.e. α is not satisfiable, which proves Gödel's theorem.

I. L. Novak (Princeton, N. J.).

Horn, Alfred. On sentences which are true of direct unions of algebras. J. Symbolic Logic **16**, 14–21 (1951).

The author studies the question of what types of sentence are necessarily true in a direct union of algebras, if true in all the factor algebras, or in at least one factor algebra, and the reverse question. He proves thirteen theorems, of which the following three are samples: (1) A closed sentence in prenex normal form whose matrix is a conjunction of equations is true in the direct union if and only if it is true in each factor algebra. (2) A closed sentence in prenex normal form whose matrix is a disjunction of inequalities is true in the direct product if and only if it is true in at least one factor algebra. (3) Given a nontautologous formula P of the propositional calculus which has no disjunctive normal form without negations, and an arbitrary set Q of quantifiers for the individual variables x_1 to x_m, then there exist two similar algebras and a sentence T formed by substituting for the propositional variables of P, equations in the individual variables x_1 to x_m, and then applying the quantifiers of the set Q, such that the sentence T is true in the direct union of the two algebras, but false in one of them. *O. Frink.*

Henkin, Leon. An algebraic characterization of quantifiers. Fund. Math. **37**, 63–74 (1950).

This deals with extensions of algebraic models of various propositional calculi, such as the classical, intuitionistic, and modal logics, to the corresponding first-order functional calculi of propositional functions and quantifiers. It is a generalization in two directions of work of Mostowski [J. Symbolic Logic **13**, 204–207 (1948); these Rev. **10**, 421]. Instead of making separate extensions for the different calculi, the author considers a rather primitive propositional calculus called the system of positive implicative logic. This has no negation, only one operation (implication), two axioms, $A\supset(B\supset A)$ and $(A\supset(B\supset C))\supset((A\supset B)\supset(A\supset C))$, and one rule, modus ponens. The formulas of this calculus are common to all the propositional logics considered.

The corresponding algebraic model, called an implicative model, has just one operation, subtraction, and a special zero element. Such a system is always a partially ordered set, and in important special cases is a complete lattice. This basic logic is then extended to a functional calculus by the addition of individual variables, function variables, and quantifiers, with the proper axioms. At the same time the algebraic interpretation is extended in the manner used by Mostowski, except that in the interpretation of the universal quantifier, the lattice operation sup Y is replaced by a slightly different operation called top Y. It is then shown that a well-formed formula of the author's functional calculus is a formal theorem if and only if it is valid in every algebraic model. Mostowski proved this in one direction only and for the intuitionistic calculus only. By the addition of operations and axioms it is then shown how the familiar functional calculi may be obtained. *O. Frink.*

L'Abbé, Maurice. On the independence of Henkin's axioms for fragments of the propositional calculus. J. Symbolic Logic **16**, 43–45 (1951).

It is shown that the axiom-schemata $A\supset.B\supset A$ and $A\supset B\supset.A\supset(B\supset C)\supset.A\supset C$ of Henkin's axioms for fragments of the propositional calculus which include material implication [cf. L. Henkin, J. Symbolic Logic **14**, 42–48 (1949); these Rev. **11**, 487] are independent of the remaining axioms of the system $S(\supset,\phi)$ for all ϕ, whereas the third schema $A\supset C\supset.A\supset B\supset C\supset C$ is independent in the systems $S(\supset)$ and $S(\supset,f)$ and dependent in $S(\supset,\sim)$.

I. L. Novak (Princeton, N. J.).

Moh, Shaw-Kwei. The deduction theorems and two new logical systems. Methodos **2**, 56–75 (1950).

Let a formal system S be given, in which a binary operator C occurs. The author investigates different forms of the deduction theorem for S, e.g. (T_1): "If p is provable in S from the suppositions $s_1,\cdots,s_n$ and if no variables in the suppositions and in the axioms of S are changed in the proof, then $Cs_1\cdots Cs_np$ is asserted in S." (T_2): "Under the hypotheses of T_1, Csp is asserted in S, where s is $Ks_1\cdots Ks_{n-1}s_n$." (T_3): "Under the hypotheses of T_1, $CKstp$ is asserted in S, where s is as in T_2 and t is the conjunction of the axioms of S which appear in the proof. Necessary conditions are given for the validity in S of either of these theorems. Applications are made to the case where S is the calculus of propositions in Gentzen's form but without the "paradoxical" rule, "From p we have Cqp", and amplified by additional axioms and rules. If S satisfies T_1, S is the material system, but in the case of T_2 or T_3 different systems are possible, some of which are studied in detail; their relations to the material system and to the systems of Lewis are made clear.

A. Heyting (Amsterdam).

Rose, Alan. Completeness of Łukasiewicz-Tarski propositional calculi. Math. Ann. **122**, 296–298 (1950).

This paper is concerned with a theorem concerning the relationship between two forms of completeness for m-valued propositional calculi. The theorem proved states that for any formalized m-valued propositional calculus satisfying certain conditions, if one add an unprovable formula as an axiom, a necessary and sufficient condition that all formulae thereby become provable is that the added formula is not an identical (valid) formula of the two-valued propositional calculus.

R. M. Martin (Philadelphia, Pa.).

Rose, Alan. The degree of completeness of some Łukasiewicz-Tarski propositional calculi. J. London Math. Soc. **26**, 47–49 (1951).

The author has recently shown [see the preceding review] that if a certain type of formalization of the m-valued Łukasiewicz-Tarski logic is weakly complete, then the addition as an axiom of an unprovable formula causes all formulas to be provable if and only if the added formula is not provable in the two-valued calculus. In the present paper he supplements this result by showing that if $m-1$ is prime, then adding as an axiom to a weakly complete formalization of the m-valued calculus a provable formula of the two-valued calculus which is not provable in the original formalization, causes all provable formulas of the two-valued calculus to be provable in the new system.

O. Frink (State College, Pa.).

The review is misleading. In addition to the set of four axioms described, the author also gives the following set of three axioms: $X \vee \sim(Y \& \sim Y) \vee X$, $X \to Y \vee X$, $(Y \vee \sim X) \to (Z \vee X \to Y \vee Z)$, which he shows to be sufficient. The resulting system is considerably simpler than Wajsberg's.

Rose, Alan. A lattice-theoretic characterisation of three-valued logic. J. London Math. Soc. **25**, 255–259 (1950).

The author gives an interpretation of the Łukasiewicz three-valued logic in which the elements are ordered pairs (p, q) of elements of a Boolean algebra such that $p \leqq q$. Using the Slupecki formalization of the three-valued logic in terms of a special implication operation Cpq and two unary operations Np and Rp, he defines these operations in his interpretation to be: $C(p, q)(r, s) = (p' \cup r, p' \cup s)$, $N(p, q) = (p', p')$, $R(p, q) = (p' \cap q, q)$. He shows that Slupecki's nine axioms hold. Hence this interpretation characterizes the three-valued logic as a formal system, since the latter is strongly complete. The intuitive meaning of the formulas in this interpretation is illustrated by a kind of Venn diagram. The only lattice theory that appears is the use of Boolean algebra. *O. Frink* (State College, Pa.).

Rose, Alan. Axiom systems for three-valued logic. J. London Math. Soc. **26**, 50–58 (1951).

Wajsberg has given a formalization of the Łukasiewicz three-valued logic, consisting of two rules of procedure and four axioms, stated in terms of the operations of implication and negation. The author first replaces Wajsberg's four axioms by four similar ones, and proves in thirty-nine steps that the two formalizations are equivalent. He then shows that if his new fourth axiom is slightly modified, his third axiom may be replaced by a simpler one. He ends up with the following four axioms: $(X\&X) \vee X \to X$, $X \to Y \vee X$, $X \to X$, and $(X \to Y) \to (Z \vee X \to Y \vee Z)$. The operations $X\&Y$ and $X \vee Y$ which appear here are definable in terms of implication and negation. The net result is a formal system similar to Wajsberg's, but slightly simpler. *O. Frink.*

Hermes, Hans. Zur Theorie der aussagenlogischen Matrizen. Math. Z. **53**, 414–418 (1951).

The author gives a proof of a theorem of Lindenbaum concerning the truth table method. This theorem was first announced in 1930 in a paper by Łukasiewicz and Tarski [C. R. Soc. Sci. Varsovie **23**, 30–50 (1930)], but no previous proof seems to have appeared. The theorem states that corresponding to every system S of formulas of the propositional calculus which is closed under the rules of substitution and modus ponens, there exists a countable normal logical matrix which is adequate for the system. The formulas of S are expressible in terms of five operations corresponding to negation, implication, equivalence, disjunction, and conjunction. The matrix is a set of truth tables for these operations, in terms of a countable set of truth values, some of which are "designated." In his proof the author takes his set of truth values to be the set of all well-formed formulas of the calculus, the designated values being the formulas of the given system S. He points out that the theorem may be generalized by replacing the five operations by others, and replacing modus ponens by other rules of procedure.
O. Frink (State College, Pa.).

Kalicki, Jan. Note on truth-tables. J. Symbolic Logic **15**, 174–181 (1950).

The author introduces and studies an algebra whose elements are truth tables determining a binary connective Δ. The truth tables involve a finite number of truth values, some designated and some undesignated. Corresponding to each truth table M he considers the set of all formulas formed from variables and the connective Δ, which are tautologies for M. The product $A \cdot B$ of two truth tables A and B is a truth table formed in a particular manner from A and B and such that the set of tautologies corresponding to it is the intersection of the tautology sets for A and B. Likewise the sum $A+B$ of truth tables is defined. Its tautology set is the union of the tautology sets of A and B. Also relations of isomorphism, equality, and inclusion of truth tables are defined, and particular truth tables O and U are introduced which play the role of zero and unit elements of the algebra. The resulting system resembles a lattice. The formal properties of this system are not completely worked out. Examples are given for special cases of the problem of determining whether the tautology set for a given truth table is empty or not. *O. Frink.*

Kalicki, Jan. A test for the existence of tautologies according to many-valued truth-tables. J. Symbolic Logic **15**, 182–184 (1950).

It is shown that there is an effective procedure for deciding whether the set of tautologies determined by a given truth table with a finite number of truth-values is empty or not [see the preceding review]. The author also defines the notion of a cyclic truth table, and proves that a cyclic truth table has no tautologies if there are any undesignated elements. He raises the question whether every finite truth table with no tautologies is either cyclic, or obtainable from a cyclic table by operations of extension and repetition which he has defined, but he does not solve this problem.
O. Frink (State College, Pa.).

Martin, Gottfried. Ueber ein zweiwertiges Modell einer vierwertigen Logik. Methodos **1**, 385–389 (1949).

The author constructs a simple, set-theoretical interpretation for four-valued propositional logic. *R. M. Martin.*

Rieger, Ladislav. On the lattice theory of Brouwerian propositional logic. Acta Fac. Nat. Univ. Carol., Prague no. **189**, 40 pp. (1949). (English. Czech summary)

The author obtains metalogical theorems about the Heyting sentential calculus by means of the theory of special distributive residuated lattices with unit and zero, which he calls "sdruz-lattices." Several of these results were established earlier by McKinsey and Tarski [J. Symbolic Logic **13**, 1–15 (1948); these Rev. **9**, 486] by means of the dual notion of a Brouwerian algebra, but are here proved independently. (In a footnote it is stated that the author did not see the paper of McKinsey and Tarski until his own work was ready for press.) It is shown that, if I is any additive ideal in a sdruz-lattice L, and if we write $x \equiv y$ whenever there exists a c in I such that $x \cap c = y \cap c$, then the relation $\equiv$ is a congruence over L; thus L/I is a homomorphic image of L; and conversely, every homomorphic image of L can be obtained in this way from some additive ideal I. If I is the additive ideal consisting of the elements x of L such that $n : x = n$ (where n is the zero-element of L, and ":" stands for the operation of residuation), then L/I is a Boolean algebra. A lattice L, with a set of generators G, is said by the author to be free if, for every lattice L' with generators G', every mapping of G into G' can be extended to a homomorphic mapping of L into L'; the author had apparently not seen the 1948 edition of Garrett Birkhoff's Lattice Theory [Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed.; these Rev. **10**, 673], where the analogous definition is given for free algebras in general (from a remark on page 5 of the paper under review, it appears that the author has been handicapped by the difficulty of obtaining mathematical literature in Czechoslovakia). It is shown that, by

considering well-formed formulas of the Heyting calculus as equivalent when they give the same value for every finite sdruz-lattice, we obtain a free sdruz-lattice with a countable infinity of generators; and the latter sdruz-lattice is shown to be a characteristic matrix for the Heyting calculus, because it can also be obtained by considering well-formed formulas α and β of the Heyting calculus as equivalent if both $\alpha \rightarrow \beta$ and $\beta \rightarrow \alpha$ are provable.

A decision method for the Heyting calculus is given, which is more explicit than that given by McKinsey and Tarski [loc. cit.], though equally impracticable to apply. A number-theoretic realization is given for the free sdruz-lattice with one generator. From the fact that this lattice is infinite, it follows that the Heyting calculus contains an infinity of nonequivalent well-formed formulas involving any given variable.

The author proves a theorem of Gödel (that, whenever $\alpha \vee \beta$ is provable in the Heyting calculus, then either α is provable or β is provable) in the following interesting lattice-theoretic form: if u is the unit element of the free sdruz-lattice L with a countable infinity of elements, and if I is the set of all elements of L different from u, then I is a multiplicative ideal. At the end of the paper the author presents two new ways of formulating the Heyting calculus: the first of these is based on the metalogical notion, "— has the same meaning as . . .", and the second on the metalogical notion "— has as a consequence . . .".

J. C. C. McKinsey (Stanford, Calif.).

Copi, Irving M. The inconsistency or redundancy of Principia Mathematica. Philos. and Phenomenol. Res. 11, 190–199 (1950). (English. Spanish summary)

This paper is concerned with the troublesome Axiom of Reducibility connected with Russell's ramified theory of types. The author claims that if the ramified theory is interpreted as containing semantics or is formulated as containing some semantical primitives, then a form of the Grelling paradox (concerning "heterological") can be derived in it. And on the other hand, if the ramified theory is not regarded as containing semantics, then the contradictions which it is supposed to avoid are already avoided anyhow. Such semantical paradoxes would simply not be expressible in the system, so that in this case the complications of type theory due to ramification would be redundant.

R. M. Martin (Philadelphia, Pa.).

Schröter, Karl. Der Nutzen der mathematischen Logik für die Mathematik. Arch. Math. Logik Grundlagenforsch. 1, 2–16 (1950).

The author gives examples to show that it is desirable to use symbolic logic and its notation in mathematics classes on an elementary and intermediate level, and in particular in teaching mathematical induction, differential and integral calculus, the foundations of the real number system, and abstract algebra. *O. Frink* (State College, Pa.).

Borgers, Alfons. Development of the notion of set and of the axioms for sets. Synthèse 7, 374–390 (1949).

This paper is expository and historical. The author describes the development of the original Cantor set theory, Zermelo's axiomatization of set theory, and the precise form of the Zermelo set theory due to Skolem. Finally, he sketches the contributions of Fraenkel and von Neumann. [Errata: p. 377, l. 4, the "y" should be "Y" in all three occurrences; p. 382, l. 20, delete the fourth right parenthesis; p. 384, l. 12, delete the fourth right parenthesis.]

R. M. Martin (Philadelphia, Pa.).

Wang, Hao. Set-theoretical basis for real numbers. J. Symbolic Logic 15, 241–247 (1950).

In a previous paper [J. Symbolic Logic 13, 129–137 (1948); these Rev. 10, 229] the author has described a logical system L which is a modification of a system of W. V. Quine. He now shows how the theory of real numbers may be developed in L. Real numbers are classes of ordered pairs of relatively prime integers. A procedure is outlined for proving in L nineteen of twenty axioms of Tarski for the real numbers as elements of a complete ordered field. The twentieth axiom, to the effect that every bounded class of real numbers has a least upper bound, is provable only in a weakened form, since not every class of real numbers is nameable in L. Proofs are omitted, but they are to be found in the author's dissertation, of which this paper is a partial summary. *O. Frink* (State College, Pa.).

Goodstein, R. L. Mean value theorems in recursive function theory. I. Differential mean value theorems. Proc. London Math. Soc. (2) 52, 81–106 (1950).

This is a contribution to recursive analysis, i.e. to that modification of rigorous analysis where the existential aspect of the "for every ϵ there exists a δ (or an N)" is replaced by a constructive argument involving recursive functions. Here the ϵ is always 10^{-k}, and the δ (or N) is a recursive function of k and various parameters; rational functions are derived from numerical functions much as rational numbers are derived from the natural integers; and real functions are replaced by sequences $f(n, x)$ of rational functions which are convergent in the modified sense. Continuity and differentiability are so defined as to entail uniformity. The author derives here analogues of the basic theorems of the differential calculus, including Rolle's theorem, the law of the mean, Cauchy's theorem (related to l'Hôpital's rule) and the Taylor theorem. *H. B. Curry* (State College, Pa.).

Heyting, A. Note on the Riesz-Fischer theorem. Nederl. Akad. Wetensch. Proc. Ser. A. 54 = Indagationes Math. 13, 35–40 (1951).

Because it is an existence theorem, the Riesz-Fischer theorem is interesting from the point of view of the intuitionists of Brouwer's school, with their requirements of constructiveness in proofs. The author gives a proof of the intuitionist version of the theorem by modifying at two stages a nonintuitionistic proof of von Neumann. The content of the resulting theorem is somewhat different from that of the usual Riesz-Fischer theorem; the hypothesis and conclusion are both stronger than usual, since they require the measurability of the functions involved in the sense of Brouwer rather than of Lebesgue. *O. Frink.*

Dugué, Daniel. L'infini en logique et les éléments définis et non calculables. Bull. Soc. Roy. Sci. Liège 19, 490–499 (1950).

***Carnap, Rudolf. Logical Foundations of Probability.** The University of Chicago Press, Chicago, Ill., 1950. xvii+607 pp. $12.50.

The book under review is the first volume of a projected two-volume work on the foundations of probability. The present volume includes the building of the formal foundations with the aid of logic and semantics. The influence of the author's philosophical position on this construction is

very great. However, we shall not discuss it in this review beyond remarking that it seems to tend more and more toward idealism.

The author distinguishes two groups of problems regarding probability: those concerning probability as degree of confirmation (probability$_1$) and those concerning probability as relative frequency (probability$_2$) [cf. Carnap, Philos. and Phenomenol. Res. **5**, 513–532 (1945); these Rev. **7**, 189]. The first of these is discussed mainly, and the author distinguishes further the problems of probability$_1$ as a quantitative concept and the problems of probability$_1$ as a comparative concept. The theory of both concepts is developed on the basis of the same language systems, their syntax and semantics being developed in detail [cf. Carnap, Philos. Sci. **12**, 72–97 (1945); these Rev. **7**, 46].

The problems of probability as a comparative concept concerns the tetradic relation between the sentence $\mathfrak{ME}(h, e, h', e')$ which means "h is confirmed by e equally or more strongly than h' by e'". The problems of quantitative concept concern the real function $c(h, e) = p$, i.e., "e confirms h with probability p".

In carrying through the analysis of the concept of the degree of confirmation the author imposes more and more severe conditions which the function $c(h, e)$ has to fulfill. The very last result is the finding of precisely one function [$c^*(h, e)$; cf. the 2d cited paper] which is, in meaning of the author, the adequate explicatum for the concept of degree of confirmation. The conviction that only one function can be an adequate explicatum for this concept follows from the point of view adopted by the author that the theory of probability is a logic of induction whose theorems are just as aprioristic as the theorems of deductive logic.

Other problems more or less closely connected with probability are also discussed by the author (Chapter I: On explication, Chapter VI: Relevance and irrelevance, Chapter IX: Estimation). *J. Łoś* (Wrocław).

ALGEBRA

Sade, Albert. An omission in Norton's list of 7×7 squares. Ann. Math. Statistics **22**, 306–307 (1951).

The author's enumeration of Latin squares of side 7 [private publication, Marseille, 1948; these Rev. **10**, 278] gave 14,112 more reduced squares than that of Norton, but he then thought it did not increase Norton's number, 146, of species. This is now abandoned; the 14,112 extra squares belong to a new species, a prototype of which is exhibited. *J. Riordan* (New York, N. Y.).

Nair, K. R. Partially balanced incomplete block designs involving only two replications. Calcutta Statist. Assoc. Bull. **3**, 83–86 (1950).

R. C. Bose [Sankhya **4**, 323–338 (1938), p. 337] showed that some designs of the type mentioned in the title may be obtained from square lattices and by inversion from some incomplete balanced block designs. The author notes that Harshbarger's [Virginia Agricultural Experiment Station, Memoir 1 (1947); these Rev. **10**, 202] rectangular lattices also belong to this type. He also adds a new method of construction for the case that the number of varieties is of the form p^2q (p, q integral). In all, 28 designs of this type are listed. *H. B. Mann* (Columbus, Ohio).

Tietze, Heinrich. Über gewisse Umordnungen von Permutationen und ein zugehöriges Stabilitäts-Kriterium. II. Jber. Deutsch. Math. Verein. **54**, Abt. 1, 63–96 (1951).

The second part of a more detailed presentation of material already reviewed [cf. S.-B. Math. Nat. Abt. Bayer. Akad. Wiss. **1943**, 135–148 (1944); these Rev. **8**, 127]. *J. Riordan* (New York, N. Y.).

Grossman, Howard D. Fun with lattice points. Scripta Math. **16**, 207–212 (1950).

Suppose an election results in km votes for A and kn for B. In how many orders may votes be cast so that A's vote is always at least m/n times B's? The author gives without proof the formula

$$p_k = \sum F_1^{k_1} F_2^{k_2} \cdots [k_1! k_2! \cdots]^{-1}$$

with $k_1+2k_2+\cdots=k$, $F_j = j^{-1}(m+n)^{-1}\binom{jm+jn}{jn}$ and the sum over all partitions of k. He also gives a short introduction to enumerations in three-dimensional lattices and derives a solution to a corresponding election problem, noting its agreement with MacMahon's. *J. Riordan.*

Bower, Julia W. An application of determinants to the probability of mated pairs. Amer. Math. Monthly **58**, 238–244 (1951).

The problème des rencontres, that is, the enumeration of permutations such that exactly x elements are in positions given by a standard order, is considered in the form: how many terms in the expansion of a determinant contain x elements from the principal diagonal, and applied to some statistical problems. *J. Riordan* (New York, N. Y.).

Rodeja F., E. G.-. Note on a lemma of A. W. Goodman. Proc. Amer. Math. Soc. **2**, 314–317 (1951).

It is proved that if F_n and G_n are arbitrary functions of the integer n then

$$\sum_{n=k}^{n} \frac{(-1)^{n+k} G_n \prod_{a=1}^{n-1} (F_n - F_a)}{G_k \prod_{a=1}^{k-1} (F_k - F_a) \prod_{a=k+1}^{n} (F_a - F_k)} = \delta_n^k.$$

The special case $G_n = n$, $F_n = n^2$, yields the lemma referred to in the title. *A. W. Goodman* (Lexington, Ky.).

Leemans, J. Sur une méthode pour obtenir les relations entre polynomes s_n donnant la somme des n^e puissances des x premiers nombres entiers. Mathesis **60**, 89–92 (1951).

Eljoseph, Nathan. On determinants with integral elements. Riveon Lematematika **4**, 22–28 (1950). (Hebrew. English summary)

Let A be a $k \times n$ matrix of integers ($k < n$), and let d be the greatest common divisor of all the k-rowed minors. It is proved that A can be augmented by $n-k$ rows in such a way as to obtain a matrix of determinant d. One proof makes use of elementary column transformations. A second proof is of geometrical nature; the reviewer was unable to follow this. It is shown how to compute all the possible ways of thus augmenting A once one way is known. *I. S. Cohen.*

Potter, H. S. A. The volume of a certain matric domain. Duke Math. J. **18**, 391–397 (1951).

Let $A < B$ mean that $B - A$ is a positive definite real symmetric matrix. Let S, T be real symmetric positive definite

matrices of orders m, n respectively. Let X be a real $m\times n$ matrix; its elements x_{ij} determine a point in Euclidean mn-space. The value of $\int|T-X'SX|^{\alpha}\{dX\}$ integrated over the domain $\mathfrak{E}(S, T)$ defined by $X'SX<T$ is determined as

$$|S|^{-\frac{1}{2}n}|T|^{\alpha+\frac{1}{2}m+\frac{1}{2}mn}\pi^{\frac{1}{2}mn}\Gamma_m(\alpha+\tfrac{1}{2}(m+1))/\Gamma_m(\alpha+\tfrac{1}{2}(m+n+1))$$

if $\alpha>-1$, where $\Gamma_m(\alpha)=\pi^{m(m-1)/4}\prod_{r=0}^{m-1}\Gamma(\alpha-\frac{1}{2}r)$. Taking $\alpha=0$ gives the volume of $\mathfrak{E}(S, T)$. For $m=n$ the result is obtained in two ways. *L. Tornheim.*

Brenner, J. L. Post-multiplication by a unitary matrix. Math. Notae **9**, 130–132 (1949).

In this article a canonical form is obtained for a matrix with complex elements under the equivalence relation of right multiplication by a unitary matrix. If A is a non-singular square matrix, then its canonical form is the (unique) positive definite Hermitian matrix P such that $A=PU$ (U unitary). If A has rank r its canonical form is a matrix $C=AU$ which has zeros in all columns beyond the first r and which has a positive definite Hermitian matrix as its "highest" nonsingular $r\times r$ minor (here "highest" refers to dictionary order on the row and column indices of the minors). *R. M. Thrall* (Ann Arbor, Mich.).

Popov, B. S. Sur une équation algébrique proposée par Pitoiset. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **37**, 17–19 (1951).

Solutions are given for algebraic equations of the form

$$x^n+a_1px^{n-2}+a_2p^2x^{n-4}+\cdots+a_kp^kx^{n-2k}+\cdots+q=0,$$

where $a_k=nC^k_{n-k}/(n-k)$, $C^k_n=\binom{n}{k}$, and p and q are any numbers. *E. Frank* (Chicago, Ill.).

***Tschebotaröw, N. Grundzüge der Galois'schen Theorie.** Übersetzt und bearbeitet von H. Schwerdtfeger. P. Noordhoff, Groningen, 1950. xvi+432 pp. 17.50 florins; bound, 20.00 florins.

This is primarily a thorough textbook for graduate students. The first 110 pages are devoted to group theory, emphasizing permutations. Then follow brief discussions of fields, symmetric functions (by a method due to Cauchy), and Galois theory by a method much closer to Galois's original one, and much longer, than most modern treatments; the Galois group of an equation is defined as the set of all permutations leaving invariant the set of relations between the roots, this set of relations being the obvious set of polynomials in several indeterminates. There is a thorough treatment, by this method, of cyclotomic extensions, constructions with ruler and compass (but not ruler and dividers), construction of extensions with the symmetric group, and Loewy's [S.-B. Heidelberger Akad. Wiss. 1925, no. 7; 1927, no. 1] theory of isomorphisms of non-normal fields. All this is restricted to finite extensions of the rational field. Galois fields are treated separately, and Galois theory over general fields is not discussed. No new theorems are proved. The reviewer disagrees with the author's contention (preface) that these methods are better training for research than a more "abstract" approach, and would not recommend this book to one learning Galois theory. But for anyone doing research on lines related to its particular approach this book is valuable and is an efficient substitute for study of the older literature.

G. Whaples (Bloomington, Ind.).

Foulkes, H. O. The new multiplication of S-functions. J. London Math. Soc. **26**, 132–139 (1951).

J. A. Todd [Proc. Cambridge Philos. Soc. **45**, 328–334 (1949); these Rev. **10**, 672] has discussed the evaluation of $\{\lambda\}\otimes S_r$ as a step towards the evaluation of the "new multiplication" or "plethysm" of S-functions. This paper concerns the evaluation of $\{1^m\}\otimes S_r$, which leads by the theorem of conjugates to an expansion for $\{m\}\otimes S_r$. The author observes that $[\{\lambda\}+\{\mu\}]\otimes S_r=\{\lambda\}\otimes S_r+\{\mu\}\otimes S_r$ and also that $S_r\otimes\{\lambda\}=\{\lambda\}\otimes S_r$. Again, if $S_r=\sum\xi_1^r$, then $\{1^m\}\otimes S_r=\sum\xi_1^r\xi_2^r\cdots\xi_m^r$. This can be converted into S-functions by a method due to the reviewer [The Theory of Group Characters and Matrix Representations of Groups, Oxford University Press, 1940, p. 89; these Rev. **2**, 3]. The author obtains in this way an explicit but rather complicated formula for all the S-functions which appear in $\{m\}\otimes S_r$. The method affords a means of classifying this usually large number of partitions into a much smaller number of sets, each set corresponding to a partition of m into no more than r parts. *D. E. Littlewood* (Bangor).

Todd, J. A. The complete irreducible system of four ternary quadratics. Proc. London Math. Soc. (2) **52**, 271–294 (1951).

Complete systems of concomitants for two and three ternary quadratics were given respectively by Gordan and Ciamberlini. Turnbull [same Proc. (2) **9**, 81–121 (1911)] gave an explicit system of 784 forms as a complete system for four ternary quadratics. In this paper the author shows that 181 forms on Turnbull's list are reducible, and that the remaining 603 forms constitute an irreducible set. The reducibility of the 181 forms is shown by the systematic use of the classical symbolic identities. To establish the irreducibility of the remaining 603 forms the method used employs the S-functional analysis introduced by the reviewer. The reviewer has analysed the invariants of ternary quadratics [ibid. **49**, 282–306 (1947); these Rev. **8**, 560]. Each invariant of total degree r in the ground forms was polarised to give an invariant linear in r ground forms and then analysed according to its behaviour when the ground forms were permuted. Each invariant was thus associated with a partition of r called the "type." A formula was given for the total number of invariants of given partial degrees in m ground forms corresponding to a given type. The author applies this method not merely to the invariants but to all types of concomitants, and in this way establishes the existence of an irreducible system containing the given 603 forms. *D. E. Littlewood* (Bangor).

Abstract Algebra

Dubreil, Paul. Relations binaires et applications. C. R. Acad. Sci. Paris **230**, 1028–1030 (1950).

La première partie contient quelques remarques sur les applications semi uniformes (=relations difonctionnelles du rapporteur [Bull. Soc. Math. France **76**, 114–155 (1948), p. 131; ces Rev. **10**, 502]). La seconde partie introduit la notion d'hexagonalité: Étant donné deux relations binaires R et S entre éléments de E, R est dite S-hexagonale lorsque $RSR\subset SRS$. Lorsque Σ est difonctionnelle partout défini, pour que $\Sigma R\Sigma^{-1}$ soit transitive, il faut et il suffit que R soit $\Sigma^{-1}\Sigma$-hexagonale. *J. Riguet* (Paris).

Dubreil, Paul. Comportement des relations binaires dans une application multiforme. C. R. Acad. Sci. Paris **230**, 1242–1243 (1950).

Comparaison des deux relations binaires R'_0 et R'_1 définies respectivement par

$$x'R_0'y' \rightleftarrows \forall x \varepsilon f^{-1}(x'),\ y \varepsilon f^{-1}(y') \quad xRy,$$
$$x'R_1'y' \rightleftarrows \exists x \varepsilon f^{-1}(x'),\ y \varepsilon f^{-1}(y') \quad xRy,$$

f étant une relation binaire quelconque d'un ensemble E sur un ensemble E' et R une relation binare sur E. Étude de la f-compatibilité d'une relation S' de E' avec R.

J. Riguet (Paris).

Riguet, Jacques. Les relations de Ferrers. C. R. Acad. Sci. Paris **232**, 1729–1730 (1951).

Following the author's previous work [Bull. Soc. Math. France **76**, 114–155 (1948); these Rev. **10**, 502], let R denote a binary relation on $E \times E$; R is called a Ferrers relation if $RR'^{-1}R \subset R$, where R' is the complement of R. Various equivalent conditions are stated; one is that E/R^{-1} be linearly ordered by inclusion. Another is that R is the union of rectangles $X_i \times Y_i$ with the $\{X_i\}$ and $\{Y_i\}$ each linearly ordered by inclusion and such that $X_i \subset X_j$ implies $Y_j \subset Y_i$. Theorem: If R is a Ferrers relation, then so is $RR'^{-1}R$, and their "difference" $R \cap (RR'^{-1}R)'$ is a difunctional relation. It is remarked that antireflexive Ferrers relations have already been studied by Wiener [Proc. Cambridge Philos. Soc. **17**, 441–449 (1914); **18**, 14–28 (1914)].

P. M. Whitman (Silver Spring, Md.).

Harary, Frank. On complete atomic proper relation algebras. J. Symbolic Logic **15**, 197–198 (1950).

The number of nonisomorphic proper relation algebras with exactly 2^m elements in which every atom is a singleton is shown to be equal to the number of distinct partitions of the positive integer m into summands which are positive squares.

B. Jónsson (Providence, R. I.).

Byrne, Lee. Short formulations of Boolean algebra, using ring operations. Canadian J. Math. **3**, 31–33 (1951).

First system of axioms: I. $(X+Y)+Z=X+(Y+Z)$. II. $(X+Y)+X=Y$. III. $X(W+YZ)=XW+Y(ZX)$. IV. $X(X+1)=Y+Y1$.

Second system of axioms:

I'. $X+X=P_1+(P_2+P_3)$. II. $(X+Y)+X=Y$.

Here

$$P_1=((A+B)+C)+(A+(B+C)),$$
$$P_2=C(F+\mathfrak{D}E)+(CF+\mathfrak{D}(EC)),$$
$$P_3=G(G+1)+(H+H1).$$

B. Jónsson (Providence, R. I.).

Sikorski, R. On an unsolved problem from the theory of Boolean algebras. Colloquium Math. **2**, 27–29 (1949).

Let $\mathbf{S}(X)$ be the Boolean algebra of all subsets of a set X. An ideal $\mathbf{I} \subset \mathbf{S}(X)$ is a principal ideal if it contains exactly those sets of $\mathbf{S}(X)$ which are subsets of some fixed set A. The quotient algebra is then isomorphic to $\mathbf{S}(X-A)$ and is complete. The author raises the question as to whether there exists an ideal $\mathbf{I}$ which is not principal such that $\mathbf{S}/\mathbf{I}$ is complete, and he proves the following theorem. Let $\mathbf{I}$ be an ideal containing all one-point sets, and let m be the least cardinal such that there exists a class $\mathbf{Y} \subset \mathbf{S}(X)$ of potency m such that $\mathbf{I}$ is formed of all subsets of sets belonging to $\mathbf{Y}$. Then, if there exists a class $\mathbf{Z}$ of potency m of disjoint sets which do not belong to $\mathbf{I}$, the quotient algebra $\mathbf{S}(X)/\mathbf{I}$ is not complete.

L. H. Loomis (Cambridge, Mass.).

Sikorski, Roman. On an ordered algebraic field. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **41** (1948), 69–96 (1950). (English. Polish summary)

This is a study of ordered rings and fields from the point of view of ordered sets. It is shown that all elements of an ordered ring have symmetrical and equal characters, and that the cofinal character is equal to the coinitial character. This latter character is called the character of the ring. If an ordered field has the character ω_μ, then every element has the character $[\omega_\mu, \omega_\mu]$.

Using Hessenberg's "natural sum" and "natural product" the author constructs a minimal ordered ring C_μ and a minimal ordered field W_μ with character ω_μ (ω_μ a regular initial ordinal). In C_μ every element has the character $[1, 1]$ and every gap has a character $[\omega_\nu, \omega_\nu]$ where $\nu < \mu$. If $\mu > 0$, then W_μ has the property (BW_μ): Every bounded ω_μ-sequence contains a convergent ω_μ-subsequence. All known ordered fields with the property (BW_μ) have potency $\aleph_\mu$. On the other hand W_0 is the field of rational numbers, and the only ordered field having the property (BW_0) is the field of real numbers whose potency is $2^{\aleph_0} > \aleph_0$.

B. Jónsson (Providence, R. I.).

McLaughlin, J. E. Projectivities in relatively complemented lattices. Duke Math. J. **18**, 73–84 (1951).

The author's main result is: if L is a simple relatively complemented lattice of finite dimension $n > 1$ and with null element z then any two points p and q in L have the property that a projectivity between the quotients p/z and q/z can be accomplished using not more than $2[\frac{1}{2}(n+1)]$ transposes. This is shown to be a best possible result (for L also upper semi-modular it is shown that 2 transposes suffice). Other theorems for such L include the following. Let a_p^k be the union of all points r such that r/z is projective to p/z by not more than $2k$ transposes. Then a_p^k is simple.

I. Halperin (Kingston, Ont.).

Kořínek, Vladimír. Lattices in which the Jordan-Hölder theorem is generally valid. Rozpravy II. Třídy České Akad. **59**, no. 23, 32 pp. (1949). (Czech)

Let S be a lattice. Let $a \geqq b$, $c \geqq d$ be elements of S and a/b, c/d the corresponding quotients (i.e. sets of all elements lying between a and b, resp. c and d). Let $a/b \gamma c/d$ mean $a = b \cup c$, $b \cap c = d$ and by duality $a/b \lambda c/d$: $b = a \cap d$, $a \cup d = c$. Further let $a/b \Gamma c/d$ mean that there exists a quotient x/y satisfying the relations: $a/b \gamma x/y \lambda c/d$ and by duality $a/b \lambda c/d$: $a/b \lambda x/y \gamma c/d$. The relations γ, λ are reflexive and transitive, the relations Γ, Λ are reflexive and symmetric. The lattice S is said to satisfy the lower prime quotient condition if the relations: a covers b, $a/b \Gamma c/d$ imply that c covers d. By duality one defines the upper prime quotient condition for S. Let S be a lattice in which all chains are finite. Then S is said to satisfy the Jordan-Hölder (J.-H.) theorem as to the relation γ, if for any two elements $a > b$ of S and any two maximal chains between them: $a = a_0 > a_1 > \cdots > a_r = b$, $a = b_0 > b_1 > \cdots > b_s = b$, one has $r = s$ and further there exists a one-to-one correspondence between the quotients a_i/a_{i+1}, b_j/b_{j+1} $(i, j = 0, \cdots, r-1)$ such that two corresponding quotients satisfy the relation $a_i/a_{i+1} \Gamma b_j/b_{j+1}$. By duality one defines the meaning of: S satisfies the J.-H. theorem as to the relation λ. The main result of the paper says that S satisfies the J.-H. theorem as to the relation γ (λ) if and only if it satisfies the lower

(upper) prime quotient condition. The author pays much attention to the lattice terminology; nevertheless he does not mention that a good deal of the fundamental Czech lattice terminology has been introduced since 1939 in the reviewer's papers on groupoids and decompositions of sets [Publ. Fac. Sci. Univ. Masaryk **1939**, no. 275; **1946**, no. 278; Rozpravy II. Třídy České Akad. **53**, no. 23 (1943); these Rev. **8**, 14, 449]. *O. Borůvka* (Brno).

Smiley, M. F. On the ideals and automorphisms of nonassociative rings. Proc. Amer. Math. Soc. **2**, 138–143 (1951).

Let M_r (M_l) denote the set of all right (left) multiplications of a nonassociative ring A, and M_r^* (M) denote the associative ring generated by M_r (by $M_r \cup M_l$). Let N_r (N) be the right ideal of M_r^* (of M) which is generated by the set of elements of M_r^* of the form $R_{xy}-R_xR_y$ (of M of the forms $R_{xy}-R_xR_y$, $L_{xy}-L_yL_x$, and R_x-L_x). If A has a unit element 1, then N_r (N) is the annihilator of 1 in M_r^* (M). The assumption of the existence of a left unit element (unit element) is replaced by the following condition U_l (U): $R_x=0$ implies that $x=0$ and $M_r \cap N_r=0$ ($M_r \cap N=0$). Let A satisfy condition U_l (U). Then the mapping

$$I \to I^* = [R_a+\nu;\ a \in I,\ \nu \in N_r\ (N)]$$

is a lattice isomorphism between the right ideal lattice (ideal lattice) of A and the lattice of all right ideals of M_r^* (M) which contain N_r (N). Also, the mapping $S \to S^*$, where $\mu S^* = S^{-1}\mu S$ for all $\mu \in M_r^*$ (M), is an isomorphism of the automorphism group of A onto the group of all automorphisms of M_r^* (M) which map each of the sets N_r and M_r (N, M_r, and M_l) onto itself. These theorems generalize certain results of the reviewer on nonassociative algebras with unit element [Bull. Amer. Math. Soc. **53**, 573–583 (1947); these Rev. **8**, 562]. *R. D. Schafer.*

Ostmann, Hans-Heinrich. Euklidische Ringe mit eindeutiger Partialbruchzerlegung. J. Reine Angew. Math. **188**, 150–161 (1950).

Let R be a Euclidean ring with value function g [see van der Waerden, Moderne Algebra, v. 1, 2d ed., Springer, Berlin, 1937, p. 60]. As usual, every element a/b of the quotient field of R can be written as an element of R plus a sum of fractions p_i/q_i, where $p_i, q_i \in R$, $(p_i, q_i)=1$, $g(p_i)<g(q_i)$, and $\prod q_i = b$ if a/b is in lowest terms. (R, g) is said to have unique partial fraction decomposition (upfd) in case the p's are uniquely determined by the q's. The author considers only g's satisfying: $g(a)<g(b) \leftrightarrow g(ac)<g(bc)$ for all $c \neq 0$, which implies van der Waerden's condition 1. He then proves (a) each of the following is necessary and sufficient for upfd: $g(x+y) \leqq \max[g(x), g(y)]$; the quotient and remainder in the division algorithm are unique. (b) If (R, g) has upfd then R is a field or the ring of polynomials in one indeterminate over a field with g a monotone transform of the ordinary degree function. (c) These polynomial rings are the only rings that have some value functions with and some without upfd. *D. Zelinsky* (Evanston, Ill.).

Nakayama, Tadasi. Generalized Galois theory for rings with minimum condition. Amer. J. Math. **73**, 1–12 (1951).

This paper combines previous work of the author on Galois theory of groups of outer automorphisms of rings with minimum condition [J. Math. Soc. Japan **1**, 203–216 (1949); these Rev. **12**, 237] and on commuter systems [Duke Math. J. **16**, 331–337 (1949); these Rev. **10**, 675]. Using a series of lemmas about more general rings, he deduces a Galois theory for primary rings which generalizes the theory for division rings found independently by Jacobson [Amer. J. Math. **69**, 27–36 (1947); these Rev. **9**, 4] and H. Cartan [Ann. Sci. École Norm. Sup. (3) **64**, 59–77 (1948); these Rev. **9**, 325]. Main theorem: Let R be a primary ring with unit element and minimum condition, N its radical, Z its center. Let Φ be a group of its automorphisms such that: (i) The subring K of R defined by $K_l = R_r\Phi \cap R_l$, (where R_r, K_l denote the rings of endomorphisms of R induced by right and left multiplications) has an independent finite basis over Z; (ii) $K \bmod K \cap N$ is a division ring; (iii) R is R_rK_l-regular; (iv) every automorphism of R contained in R_rK_l is contained in Φ; (v) if $\Phi_0 = \Phi \cap R_rK_l$ then Φ/Φ_0 is finite; (vi) there is a set $\{\rho_i\}$ of representatives of $\Phi \bmod \Phi_0$ such that the modules (R, ρ_i), as defined in the first paper cited above, satisfy the condition mentioned there. Let S be the invariant system of Φ in R. Then we have the expected dualism between inter-rings over which R is left regular and subgroups of Φ satisfying (i)-(vi). If $\Phi_0=1$ (his $\Phi=1$ on p. 11 is a misprint) the assumption of primality can be replaced by direct indecomposability.

G. Whaples (Bloomington, Ind.).

Skornyakov, L. A. Alternative fields. Ukrain. Mat. Žurnal **2**, 70–85 (1950). (Russian)

It is shown that an alternative field K of characteristic different from 2 or 3 is associative or is a Cayley-Dickson algebra over its center. This important result has been obtained independently by R. H. Bruck and Erwin Kleinfeld and presented at the International Congress of Mathematicians in September, 1950. The proof by Bruck and Kleinfeld also covers the case of characteristic 3. The two proofs are quite different. Bruck and Kleinfeld base their proof on showing that in a nonassociative alternative field every element satisfies a quadratic equation over the center. Skornyakov proceeds by showing the properties and existence of a "special triple." Elements a, b, c form a special triple if

$$ba=-ab, \quad ca=-ac, \quad cb=-bc, \quad a(bc)=-(ab)c.$$

Because of the importance of this result, Skornyakov's proof will be sketched in some detail.

The associator $[a, b, c]$ and commutator $[a, b]$ are given by the rules: $[a, b, c]=(ab)c-a(bc)$, $[a, b]=ab-ba$. The basic identities defining an alternative ring are $(ab)b=a(bb)$, $(aa)b=a(ab)$. From these follow directly the alternating properties of the associator: $[a, b, c]=-[a, c, b]=-[b, a, c]$. In addition the following relations hold, assuming the characteristic different from 2:

(1) $(ab)a=a(ba)$,

(2) $a^{-1}(ab)=b=(ba)a^{-1}$,

(3) $(ab)(ca)=a((bc)a)$,

(4) $6[a, b, c]=[[a, b], c]+[[b, c], a]+[[c, a], b]$,

(5) $[ab, c, d]-[a, bc, d]+[a, b, cd]=a[b, c, d]+[a, b, c]d$,

(6) $[ab, a, c]+[ba, a, c]-[b, a^2, c]=0$,

(7) $[ab, a, c]+[a, b, ac]=0$,

(8) $[ba, a, c]=-a[a, b, c]$,

(9) $[ab, a, c]=-[a, b, c]a$.

An element α such that $[\alpha, b, c]=0$ for every b and c is said to be in the associative center A of K. By specialization in (5) we find $[\alpha a, b, c]=\alpha[a, b, c]$, $[a, b, c\alpha]=[a, b, c]\alpha$, and

$[a, b\alpha, c]=[a, b, \alpha c]$. From these follows $\alpha[a, b, c]=[a, b, c]\alpha$. Also using (5), A is a subring of K, and using (2) even a subfield. On the other hand an element not belonging to the associative center is, by (8), the quotient of associators. Thus K is the union of A and the subfield M generated by associators, and every element of A permutes with every element of M.

For a special triple a, b, c as defined above it is shown that a^2, b^2, c^2 permute and $[a^2, b^2, c^2]=0$. Hence by the theorem of Artin a^2, b^2, c^2 generate a field. It is then shown that the alternative field $P(a, b, c)$ has a basis of eight elements $1, a, b, c, ab, ac, bc, a(bc)$ over the field $P(a^2, b^2, c^2)$ and is a Cayley-Dickson algebra C. Modifying the arguments of the preceding paragraph it is shown that $K=C\times K^C$ where K^C is the alternative subfield of elements associating and permuting with C. It is next shown that K^C is associative and using theorems on associative rings that K^C is the center of K. Thus the main result is proved if K contains a special triple.

It remains to be shown that in a nonassociative alternative field there is a special triple. If K is nonassociative, then there exist a, b, c with $u=[a, b, c]\neq 0$. Here $[a, c]u=-u[a, c]$, $[a, b]u=-u[a, b]$, and $[b, c]u=-u[b, c]$. This yields elements a_1, b_1 with $b_1a_1=-a_1b_1$ unless $[a, b]=[a, c]=[b, c]=0$. But from (4) this would imply $6[a, b, c]=0$ or $[a, b, c]=0$ contrary to assumption. (This appears to be the only place in the paper using the hypothesis that the characteristic of K is different from 3.) Next, if for fixed a, b and all t, $[a, b, t]=0$ it is shown that $[a, b]$ is in the center of K. But if $ba=-ab$, then $\frac{1}{2}[a, b]=ab$ is not in the center since $(ab)a=-a(ab)$. Thus for $ba=-ab$ there is some $c=[a, b, t]\neq 0$ and the elements a, b, c are shown to be a special triple.

As an application of the main theorem, a question raised by the reviewer is answered. A projective plane which has an alternative field of characteristic not 2 or 3 as a natural ring will have collineations transitive on its quadrilaterals and so all its natural rings will be isomorphic.

Marshall Hall (Washington, D. C.).

Skornyakov, L. A. Right-alternative fields. Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 177–184 (1951). (Russian)

A ring R is right alternative if the law $(ab)b=a(bb)$ holds. It is a field if it has no divisors of zero and equations $xa=b$, $ax=b$ have solutions. It is shown that a right alternative field of characteristic not 2 is an alternative field. This result is applied to projective planes. If the minor theorem of Desargues is valid for all choices involving one of two given lines as the axis of perspectivity, then it is a universal theorem.

Marshall Hall (Washington, D. C.).

Fuchs, Ladislas. The generalization of the valuation theory. Duke Math. J. **18**, 19–26 (1951).

In the definition of a valuation function v, the author replaces the usual triangle inequality by the condition that, if $v(a)\geqq v(c)$ and $v(b)\geqq v(c)$, then $v(a-b)\geqq v(c)$. This yields valuations with partially ordered Abelian value groups. In fact, every integral domain R (with an identity element) becomes a valuation ring of its quotient field. Various relations are established between the ideals in R and the ordering in the value group Γ; for example, Γ is Archimedian-ordered if and only if every ideal in R is primary. It is also shown that the value group of an integrally closed ring is the subdirect sum of linearly ordered groups.

B. N. Moyls (Vancouver, B. C.).

***Chatelet, Albert. L'arithmétique des idéaux.** Les Conférences du Palais de la Découverte. Université de Paris, 1950. 26 pp.

Elementary expository article. *R. Hull.*

Buck, R. Creighton. A factoring theorem for homomorphisms. Proc. Amer. Math. Soc. **2**, 135–137 (1951).

The author considers a class of groups which share a class Ω of operators α applicable to finite or transfinite sequences and such that $\alpha(0, 0, \cdots)=0$. Such systems are referred to as algebras. The main theorem is: Let $A, B_1, B_2, \cdots, B_n, C$ be algebras with the B_i simple. Let ϕ_i be a homomorphism of A onto B_i and let ψ be a homomorphism of A into C; let $K_i=\text{kernel }(\phi_i)$ and $K=\text{kernel }(\psi)$. Then, the condition $\bigcap K_i\subset K$ is necessary and sufficient for the existence of homomorphisms θ_i of B_i into C such that $\psi=\sum\theta_i\phi_i$. This result is used to give a simple proof of the Chevalley-Jacobson density theorem for irreducible rings of endomorphisms.

B. Jónsson (Providence, R. I.).

Amitsur, A. S., and Levitzki, J. Remarks on minimal identities for algebras. Proc. Amer. Math. Soc. **2**, 320–327 (1951).

This paper supplements an earlier one of the authors [same Proc. **1**, 449–463 (1950); these Rev. **12**, 155]. All minimal identities are determined for the algebra of two by two matrices over the field of two elements; apart from this exceptional case, it is shown that all minimal identities of a semi-simple algebra are linear.

I. Kaplansky.

Brown, Bailey. An extension of the Jacobson radical. Proc. Amer. Math. Soc. **2**, 114–117 (1951).

Portions of the theory of the radical in an arbitrary ring, as given by Jacobson [Amer. J. Math. **67**, 300–320 (1945); these Rev. **7**, 2], are here extended to a cluster [cf. the reviewer, Trans. Amer. Math. Soc. **63**, 482–513 (1948); these Rev. **9**, 565]. In a cluster R, an element a is called right quasi-regular provided a belongs to the right ideal generated by all elements of the form $ax-x$ for $x\epsilon R$. The right radical N of R is defined as the set of all elements a such that every element in the ideal generated by a is right quasi-regular. With the aid of Zorn's lemma, N is shown to be the intersection of all ideals of a certain maximal type. The residue class cluster R/N has zero radical. A nonzero cluster is isomorphic to a subdirect sum of primitive clusters if and only if $N=0$, where a primitive cluster is a suitable generalization of the primitive ring.

R. A. Good.

Azumaya, Gorô. On maximally central algebras. Nagoya Math. J. **2**, 119–150 (1951).

Let A be a primary algebra over a field K, Z its center, N its radical, t the order of A/N over its center. In a previous paper [Jap. J. Math. **19**, 263–273 (1948); these Rev. **11**, 316] the author and Nakayama called A maximally central if $[A:K]=t^2[Z:K]$. The present paper is devoted to a systematic study of these algebras, with the added generalization that K need only be a commutative ring with unit. This generalization is appropriate since it is desirable to study A over its center. The definitions are given in a new form. Let A' be the reciprocal algebra; then A may be regarded as an $(A\times A')$-module, and A is said to be proper maximally central if $A\times A'$ is the K-endomorphism ring of A, while A is maximally central if it is a direct sum of algebras which are proper maximally central over their respective centers. It is shown that this definition coincides with the earlier one. Two typical results are: (1) the direct

product of two proper maximally central algebras is proper maximally central; (2) A is proper maximally central if and only if A/pA is central simple over K/p for any maximal ideal p in K. Next K is specialized to be a completely primary ring (i.e., one with a unique maximal ideal p). The validity of the Hensel lemma for polynomials over K plays a decisive role, and K is called a Hensel ring if it is valid. Detailed considerations, mainly concerned with the lifting of idempotents, lead to the final theorem, a generalization of the theorem of Wedderburn-Malcev: If K is a Hensel ring, and A/N is separable over K/p, then there exists a maximally central subalgebra S with $S+N=A$, $S\cap N=pA$, and S is unique up to an inner automorphism.

I. Kaplansky (Chicago, Ill.).

Preuss, Günther, und Schmidt, Friedrich Karl. Über den Hilbertschen Irreduzibilitätssatz. Math. Nachr. **4**, 348–365 (1951).

The authors prove Hilbert's [J. Reine Angew. Math. **110**, 104–129 (1892)] irreducibility theorem (if the polynomial $F(x_1\cdots x_s; t_1\cdots t_r)$ is irreducible in $K[x_1\cdots x_s; t_1\cdots t_r]$ then there are infinitely many substitutions of elements of K for the t_i such that the resulting polynomial is irreducible) for all F separable in the x_i, over a class of fields called arithmetisch halbendlich (a.h.) and defined by axioms which are a generalization of the axioms of Artin and the reviewer [Bull. Amer. Math. Soc. **51**, 469–492 (1945); these Rev. **7**, 111] in that they are satisfied for certain infinite extensions of R. The proof is a modification of Hilbert's. The set of a.h. fields is defined in two ways, both of interest. First definition: Prime divisors and valuations are defined as in Artin and Whaples [loc. cit.]. A set $\mathfrak{V}$ of prime divisors is called a Vollmenge if it is not empty and if for every $a\in K$ with $|a|_\mathfrak{p}>1$ for some $\mathfrak{p}\in\mathfrak{V}$ there is a $\mathfrak{q}\in\mathfrak{V}$ with $|a|_\mathfrak{q}<1$. A set $\mathfrak{H}$ is called a Hauptmenge if it is not empty and if every $a\in K$ is expressible as a_1/a_2 with $|a_i|_\mathfrak{p}\leqq 1$ for every $\mathfrak{p}\in\mathfrak{H}$. A field is a.h. if it has a $\mathfrak{V}$ and an $\mathfrak{H}$ such that the complement of $\mathfrak{H}$ in $\mathfrak{V}$ is finite. Second definition (more general than the first): A subset $\mathfrak{o}\subseteq K$ is a Halbordnung (h.o.) if it is a multiplicative semigroup containing 1 and -1 and, for every positive integer m, containing a semigroup ideal $\mathfrak{o}_m\neq(0)$ such that the sum of every m elements of $\mathfrak{o}_m$ is in $\mathfrak{o}$. A generalized Hauptmenge is a set $\mathfrak{H}'$ of prime divisors such that the set $\mathfrak{o}_\mathfrak{H}$, of all $a\in K$ with value at most 1 at all $\mathfrak{p}\in\mathfrak{H}'$ is a h.o. An Ergänzungsmenge of an $\mathfrak{H}'$ is a set $\mathfrak{E}$ of valuations, independent of those in $\mathfrak{H}'$, such that 0 is the only element of $\mathfrak{o}_\mathfrak{H}$ with value less than 1 at all $\mathfrak{p}\in\mathfrak{E}$. Finally, K is a.h. in the second sense if it has an $\mathfrak{H}'$ with a finite $\mathfrak{E}$. They prove further that if K is a.h. and of characteristic p then a necessary and sufficient condition that Hilbert's theorem hold also for all $F(x,t)$ inseparable in x is that $K^{1/p}\neq K$.

G. Whaples (Bloomington, Ind.).

Dieudonné, Jean. Semi-dérivations et formule de Taylor en caractéristique p. Arch. Math. **2**, 364–366 (1950).

The problem is to determine, for each integer $m\geqq 0$, the coefficient of Y^m in $f(X+Y)$, where f is a polynomial in the ring $K[X]$ of polynomials in one indeterminate X with coefficients in a field K of characteristic $p>0$. Denoting this coefficient by $\Delta_m f$ and letting $D_k=\Delta_{p^k}$, the author shows that D_k is a semi-derivation of height k [Dieudonné, C.R. Acad. Sci. Paris **227**, 1319–1320 (1948); these Rev. **10**, 280] such that $D_k f=0$ whenever f has degree $<p^k$ and $D_k(X^{p^k})=1$; from this it is easy to deduce a formula for $D_k f$ for any $f\in K[X]$. He then solves his problem by showing that if $m=\alpha_0+\cdots+\alpha_k p^k$ is the p-adic expansion of m then $\Delta_m=(\alpha_0!)^{-1}D_0^{\alpha_0}\cdots(\alpha_k!)^{-1}D_k^{\alpha_k}$. He notes some consequences and asserts that generalization to several indeterminates and power series is easy.

E. R. Kolchin.

Okugawa, Kôtaro. Basis-theorem concerning differential polynomials. Mem. Coll. Sci. Univ. Kyoto Ser. A. **25**, 93–97 (1949).

Let P be the ring of all polynomials in indeterminates y_{ij} $(1\leqq i\leqq n,\ 1\leqq j<\infty)$ with coefficients which are complex functions of $(x, y_1, \cdots, y_n)$ analytic at $(0, 0, \cdots, 0)$, and make P into an ordinary differential ring by introducing a derivation $a\to a'$ such that

$$\varphi(x, y_1, \cdots, y_n)'=\varphi_x(x, y_1, \cdots, y_n)+\sum\varphi_{y_i}(x, y_1, \cdots, y_n)y_{i1},$$

and $y'_{ij}=y_{i,j+1}$. J. F. Ritt proved [Amer. J. Math. **60**, 535–548 (1938)] that P is a differential ring with basis theorem. The author gives a new proof of Ritt's theorem.

E. R. Kolchin (New York, N. Y.).

Theory of Groups

Dubreil-Jacotin, Marie-Louise. Quelques propriétés arithmétiques dans un demi-groupe demi réticulé entier. C. R. Acad. Sci. Paris **232**, 1174–1176 (1951).

This paper is concerned with conditions under which an analogue of the unique factorisation theorem holds in a semi-group in which a union is defined which is distributive with respect to multiplication. It is shown that necessary and sufficient conditions for such a result to hold in such a semi-group of this type which has a greatest element which is also an identity are that (i) if $a\leqq b$, i.e., if $a\cup b=b$, then $a=bc$ where $a\leqq c$ and $a\neq c$, and (ii) the ascending chain condition holds in the semi-group.

D. Rees.

***Tamari, Dov. Sur l'immersion d'un semi-groupe topologique dans un groupe topologique.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 217–221. Centre National de la Recherche Scientifique, Paris, 1950.

A topological semigroup is a topological space which is closed algebraically under a continuous, binary, associative operation, with both cancellation laws. It is regular on the right if $x, y\in S\to(\exists u, v\in S)(xu=yv)$. The equivalence relation R in $S\times S$ is defined by

$$(x, y)\sim(x', y')\leftrightarrow(\exists u, v\in S)(xu=x'v,\ yu=y'v).$$

With the appropriate operation between equivalence classes $S\times S/R$ is a group G. It is shown that G may be considered as a group of partial translations of S. This paper establishes that G is a topological group if and only if S has a right uniform structure, which is invariant on the left. This means that if $\mathfrak{U}$ is a fundamental system of entourages, then $U\in\mathfrak{U}$, $f\in G\to(\exists U'\in\mathfrak{U})((f, f)U'\subset U)$, where f is considered as a partial translation. The results are extended to topological rings.

J. E. L. Peck (Durban).

Bateman, P. T. A remark on infinite groups. Amer. Math. Monthly **57**, 623–624 (1950).

The author shows that the well-ordering principle implies immediately the following theorem: For any infinite quasi-group G there exists a one-to-one mapping $x\to\theta(x)$ of G onto itself such that $x\to\eta(x)=x\cdot\theta(x)$ is also a one-to-one mapping of G onto itself. Moreover, if a and b are any given elements

of G, we can choose θ so that $\theta(a)=b$. For earlier results see Paige [Bull. Amer. Math. Soc. **53**, 590–593 (1947); Duke Math. J. **16**, 39–60 (1949); these Rev. **9**, 6; **10**, 430].

R. D. James (Vancouver, B. C.).

Schnee, Walter. Über vollständige Aufzählung von Permutationsgruppen. Math. Nachr. **5**, 135–138 (1951).

Using elementary methods the author describes the 19 types of permutation groups on five or fewer symbols of which nine types contain only even permutations and have the orders 1, 2, 3, 4, 5, 6, 10, 12, 60 respectively, whereas the other ten each contain one of these groups as an invariant subgroup Q of index 2, and an odd permutation ρ that represents the odd coset $Q\rho$. Two of the latter types are of order 4 and two of order 6, one of each of the orders 2, 8, 12, 20, 24, 120, and none of order 10. *J. S. Frame.*

Wever, Franz. Über Regeln in Gruppen. Math. Ann. **122**, 334–339 (1950).

By a rule in a group G is meant an identical relation $w(x_1, x_2, \cdots, x_n)=1$ holding for every choice of the x_i in G. It is shown that in the free group F_n generated by $x_1, \cdots, x_n$ $w(1, x_2, \cdots, x_n)=w(x_1, 1, \cdots, x_n)=\cdots=w(x_1, x_2, \cdots, 1)=1$, then w belongs to the nth commutator subgroup of F_n. Thus the rules $w=1$ for a group G may include one rule $x^m=1$ and for any others we may take w as a commutator form. The Burnside groups $B_{k,m}$ are the most general groups generated by k elements with the rule $x^m=1$. Little is known about these for $m\geqq 5$. The author studies several special rules which are commutators. He shows that if the square (cube) of any element of G is in the center, then G is nilpotent of index three (five). Also the rule $((x, y), y)=1$ implies that G is nilpotent of index $k+1$ if G is generated by k elements. *Marshall Hall* (Washington, D. C.).

Meier-Wunderli, H. Metabelsche Gruppen. Comment. Math. Helv. **25**, 1–10 (1951).

A determination of the order of the free metabelian group of prime exponent p (i.e. satisfying $x^p=1$) with n generators. The factor group of the wth group of the lower central series of the free metabelian group with n generators by the $(w+1)$th is a free abelian group of dimension

$$d_w=(w-1)\binom{n+w-2}{w}$$

if $w>1$. The effect of the identical relation $x^p=1$ is first, to replace the free abelian groups by elementary abelian p-groups, second to reduce the dimension for $w=p$ by $\binom{n+p-1}{p}-n$, and third to reduce the dimension for $w<p$ to zero. Thus the required order is p^N where

$$N=d_1+d_2+\cdots+d_w-\binom{n+p-1}{p}.$$

G. Higman (Manchester).

Neumann, Bernhard H., und Neumann, Hanna. Zwei Klassen charakteristischer Untergruppen und ihre Faktorgruppen. Math. Nachr. **4**, 106–125 (1951).

A normal subgroup $\mathfrak{H}$ of a group $\mathfrak{G}$ is said to be overcharacteristic (ultracharacteristic) in $\mathfrak{G}$ if, given any normal subgroup $\mathfrak{K}$ of $\mathfrak{H}$, the condition $\mathfrak{G}/\mathfrak{K}\cong\mathfrak{G}/\mathfrak{H}$ implies that $\mathfrak{K}\supseteq\mathfrak{H}$ ($\mathfrak{K}\subseteq\mathfrak{H}$). Several properties of these two concepts are obtained. Thus every ultracharacteristic subgroup of $\mathfrak{G}$ is overcharacteristic in $\mathfrak{G}$, and every overcharacteristic subgroup of $\mathfrak{G}$ is characteristic in $\mathfrak{G}$, the relation of being ultracharacteristic is transitive, and the intersection of arbitrarily many overcharacteristic (ultracharacteristic) subgroups of $\mathfrak{G}$ is overcharacteristic (ultracharacteristic) in $\mathfrak{G}$.

Let F be a free group with n generators $\mathfrak{x}=(x_1, \cdots, x_n)$, and let $\mathfrak{G}$ be a group with n generators. Denote by A and B the groups of automorphisms of $\mathfrak{F}$ and $\mathfrak{G}$ respectively, and let Γ be the set of all n-termed sequences $\mathfrak{g}=(g_1, \cdots, g_n)$ whose terms generate $\mathfrak{G}$. To each $\mathfrak{g}\in\Gamma$ corresponds a homomorphism $\varphi_{\mathfrak{g}}$ of $\mathfrak{F}$ onto $\mathfrak{G}$ which maps x_i into g_i. Each $\beta\in B$ may be regarded as a permutation of Γ, and to each $\alpha\in A$ we correlate a permutation of Γ by letting $\mathfrak{g}\alpha=x\alpha\varphi_{\mathfrak{g}}$. Thus A and B together generate a group P of permutations of Γ. Let $\mathfrak{R}(\mathfrak{g})$ be the kernel of $\varphi_{\mathfrak{g}}$ and let $\mathfrak{T}(\mathfrak{g})$ ($\mathfrak{U}_n(\mathfrak{G})$) be the intersection of all groups $\mathfrak{R}(\mathfrak{g}')$ with $\mathfrak{g}'\in\mathfrak{g}P$ ($\mathfrak{g}'\in\Gamma$). Then $\mathfrak{T}(\mathfrak{g})$ ($\mathfrak{U}_n(\mathfrak{G})$) is characteristic (overcharacteristic) in $\mathfrak{F}$. If $\mathfrak{G}$ is Abelian, then $\mathfrak{T}(\mathfrak{g})=\mathfrak{U}_n(\mathfrak{G})$; if $\mathfrak{G}$ is finite, then $\mathfrak{U}_n(\mathfrak{G})$ is ultracharacteristic in $\mathfrak{F}$. It is shown how $\mathfrak{F}/\mathfrak{T}(\mathfrak{g})$ and $\mathfrak{F}/\mathfrak{U}_n(\mathfrak{G})$ can be computed, and the computations are carried out in three specific cases. Several unsolved problems are mentioned. *B. Jónsson* (Providence, R. I.).

Ore, Oystein. Some remarks on commutators. Proc. Amer. Math. Soc. **2**, 307–314 (1951).

It is proved that in a symmetric group (group of all one-to-one correspondences of a set, finite or infinite, with itself) all elements of the derived group are actually commutators. The proof is by cases, the various relevant combinations of cycles being dealt with in turn.

G. Higman (Manchester).

Scott, W. R. Algebraically closed groups. Proc. Amer. Math. Soc. **2**, 118–121 (1951).

Call a group G algebraically closed (or a.c.) if every finite system of equations $W_i=1$ and inequalities $W_j\neq 1$, where W_i are words in the elements of G and in indeterminates $x_1, \cdots, x_s$, which has solutions in any group containing G has solutions in G itself. The author proves that any group can be embedded in an a.c. group; and that an a.c. group contains an isomorphic copy of every finite group.

G. Higman (Manchester).

Kontorovič, P. G. Invariantly covered groups. II. Mat. Sbornik N.S. **28(70)**, 79–88 (1951). (Russian)

The concept of A-group in part I [Mat. Sbornik N.S. **8(50)**, 423–436 (1940); these Rev. **2**, 213] is generalized by introducing the class of groups with category. Denote by $A^{(1)}$ the class of all Abelian subgroups and by $A^{(\eta)}$ the class of all groups representable as a set-theoretic union of normal subgroups each belonging to some preceding class. A member of any class $A^{(\eta)}$ is called a group with category. A group G is said to have category η if G belongs to $A^{(\eta)}$ but to no $A^{(\zeta)}$ for $\zeta<\eta$. Various basic properties are derived concerning subgroups, quotient-groups, and direct products of groups with category. For arbitrary group G and arbitrary natural number n, the nth hypercenter of G is a group with category not exceeding n, whence a group with a central series of finite length has category. The class of groups with category and the class of groups with ascending central series are shown to be closely interrelated, although neither includes the other. All periodic elements in a group with category form a subgroup. A finite group is special if and only if it has category. Finally certain relationships involving higher commutator elements are obtained in a group with finite category. *R. A. Good* (College Park, Md.).

Golovin, O. N. Nilpotent products of groups. Mat. Sbornik N.S. **27**(69), 427–454 (1950). (Russian)

The author continues his study of a variety of products of subgroups and supplies proofs and extensions of announced results [Doklady Akad. Nauk SSSR (N.S.) **58**, 1257–1260 (1947); these Rev. **9**, 493; we shall refer to this paper as O]. He extends work of (1) Baer [Trans. Amer. Math. Soc. **58**, 348–389 (1945); these Rev. **7**, 372] on minimal central and lower central chains to various kinds of higher commutators and of (2) F. W. Levi, [J. Indian Math. Soc. (N.S.) **8**, 78–91 (1944); these Rev. **7**, 113] on S-products. Part I redevelops well-known results on commutators in a form suitable for use in the remainder of the paper. If the A_α form a class of subgroups of a group G, then (A_α) represents the subgroup generated by all commutators (a_α, a_β), where $a_\alpha \varepsilon A_\alpha$, $a_\beta \varepsilon A_\beta$, and $\alpha \neq \beta$. For a subgroup A, let $\bar{A}$ be its normalizer. In part II, he carries over the definition of regular product, as given in O, in the obvious way to the case of more than two factors, along with the representation of an element of such a product G as $a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_n} u$, where $a_{\alpha_i} \varepsilon A_{\alpha_i}$, $u \varepsilon (\bar{A}_\alpha)$. In fact, $G/(\bar{A}_\alpha)$ turns out to be isomorphic to the direct product of the A_α. Also, G is a homomorphic image of the free product of the same A_α with kernel included in $(\bar{A}_\alpha)$ and conversely. For instance, if the free product of the A_α is reduced modulo the $(k+1)$st member of the minimal central chain for (A_α) in the free product, then $\Pi^{(k)} A_\alpha$, the resulting object, is a regular product of the A_α and is called the kth nilpotent (in O, central) product of the A_α. If $k \geqq l$, then one can map $\Pi^{(k)}$ onto $\Pi^{(l)}$ homomorphically. Such nilpotent products are commutative and associative in a very general sense. An example of the type of theorem in the concluding section is the following: If each A_α in $G = \Pi^{(k)} A_\alpha$ is periodic, then G is periodic, and p is a prime which divides the order of an element of G if, and only if, it divides the order of some element in some A_α. *F. Haimo* (St. Louis, Mo.).

Fröhlich, A. The representation of a finite group as a group of automorphisms on a finite Abelian group. Quart. J. Math., Oxford Ser. (2) **1**, 270–283 (1950).

Investigations on the class-groups of self-conjugate algebraic number fields have led the author to consider the problem in what manner a given finite group G of order h can be represented as a group of automorphisms of an Abelian group A of order n. This problem is solved in the present paper under the restriction that $(h, n) = 1$. The group A is written additively and so the problem immediately reduces to the investigation of automorphisms of a module of type $(p^{n_1}, p^{n_2}, \cdots, p^{n_m})$, p a prime. The main tools are Shoda's theory of the ring of endomorphisms of a finite Abelian p-group [Math. Ann. **100**, 674–686 (1928)] and an adaptation of Schur's lemma. A complete parallelism is exhibited between representations of G in the residue class ring (mod p^r), $r = \max s_i$, and in the Galois field $GF(p)$, but R. Brauer's work on modular representations is not used explicitly. Indeed the whole discussion is straightforward and elementary. The results of the author overlap in part with those obtained by G. Szekeres [Trans. Amer. Math. Soc. **66**, 1–43 (1949); these Rev. **11**, 320]: the latter restricts G to be cyclic, but allows a square-free g.c.d. (h, n).

K. A. Hirsch (London).

Nakayama, Tadasi, and Osima, Masaru. Note on blocks of symmetric groups. Nagoya Math. J. **2**, 111–117 (1951).

In this paper an alternative proof is given of the conjecture of Nakayama that two representations $[\alpha]$ and $[\alpha']$ of the symmetric group S_n belong to the same p-block if and only if they have the same p-core [cf. Brauer, Trans. Roy. Soc. Canada. Sect. III. (3) **41**, 11–19 (1947); Robinson, ibid. **41**, 20–25; these Rev. **10**, 678]. The methods are similar to those of Nakayama's first two papers on the subject [Jap. J. Math. **17**, 165–184, 411–423 (1947); these Rev. **3**, 195, 196; **4**, 340]. The important result that a diagram $[\alpha]$ exists and is completely determined by its p-core and its star diagram, one residue (mod p) being associated with each of the p disjoint constituents of the star diagram, is stated without proof. The relation of this result to the partition function is to be the subject of a later paper.

G. de B. Robinson (Toronto, Ont.).

Stanton, R. G. The Mathieu groups. Canadian J. Math. **3**, 164–174 (1951).

Character tables for the simple Mathieu groups $\mathfrak{M}_{12}$ of order $m_{12} = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8$ and $\mathfrak{M}_{24}$ of order

$$m_{24} = 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 48$$

have been published by Frobenius [S.-B. Preuss. Akad. Wiss. **1904**, 557–571, pp. 568, 570]. The main theorem of the present paper states that there are no other simple groups of the orders m_{12} and m_{24}. The proof is based on Brauer's theory of modular characters and falls into three parts. The seven lemmas in the first part describe relationships in blocks of modular characters, especially in the so-called first blocks $B_1(p)$ that contain the 1-representation. For instance, let a group of order g have elements of prime orders p and p', but not of order pp' (where $(pp', g/pp') = 1$). Also let a_{ij} denote the number of characters in $B_1(p) \cap B_1(p')$ whose degrees are $\equiv (-1)^i \pmod{p}$ and $\equiv (-1)^j \pmod{p'}$ plus the number of exceptional p-conjugate or p'-conjugate characters of type ij. Then the block intersection theorem of lemma 6 states that $a_{00} + a_{11} = a_{01} + a_{10}$. The application of these lemmas to the Mathieu groups is sketched in the second part of the paper, but the details of computation which were included in the author's University of Toronto thesis are omitted. All 26 characters for a simple group of order m_{24} are included in the $1+1+3+3$ blocks of lowest type for the primes 23, 11, 7, and 5, and the character table for such a group is uniquely determined. For instance, applying lemma 6 to two of these primes, it is clear that each pair of first blocks must contain at least one character of type (1, 0) or (0, 1), such as the two characters of degree 990 for $p = 23$ and $p' = 7$, or the character of degree 5796 for $p = 5$, $p' = 11$. The proof is completed in part three, when the author shows that only one group belongs to each of the character tables for $\mathfrak{M}_{12}$ and $\mathfrak{M}_{24}$. Here again the details are omitted. *J. S. Frame* (Lansing, Mich.).

Chevalley, C., and Kolchin, E. Two proofs of a theorem on algebraic groups. Proc. Amer. Math. Soc. **2**, 126–134 (1951).

Polynomial functions of m arguments $x_1, \cdots, x_m$ in an n-dimensional vector space V over an infinite field K (i.e., polynomial functions of mn coefficients of $x_1, \cdots, x_m$) form a ring o_m. Let G be a group of linear transformations of V, i.e., $n \times n$ matrices in K. An element of the quotient field R_m of o_m is an invariant (semi-invariant) if it is left fixed (multiplied by scalars) under G. The paper proves that every algebraic group G (defined by a system of polynomial equations for the (matric-) coefficients) is characterized by a finite set of its invariants in R_n, or by a finite set of its semi-invariants in o_n. In fact, the second half follows easily from the first.

Two different proofs are given. In the first proof, by A_r is denoted the totality of elements in o_n with degrees $\leqq r$ which are annulled by the defining equations of G. If r is high enough, the elements of G are characterized by the property that they map A_r into itself. On considering the exterior algebra over the vector space of elements of o_n with degrees $\leqq r$, this is shown in turn to be equivalent to the multiplication of $u_1u_2\cdots u_b$ by scalars, where $\{u_1, \cdots, u_b\}$ is a basis of A_r. This can then be interpreted in terms of invariants. The second proof depends on the Picard-Vessiot theory [Kolchin, Ann. of Math. (2) **49**, 1–42 (1948); these Rev. **9**, 561], which characterizes G by means of a system of differential invariants (of n variables). It is shown that their orders may be taken $\leqq n-1$, which gives invariants in R_n.

It is proved further that for a distinguished (i.e., normal) algebraic subgroup H of an algebraic group G there exists a faithful representation of the factor group G/H, by making use of polynomial semi-invariants of H. *T. Nakayama.*

Aczél, J. Über einparametrige Transformationen. Publ. Math. Debrecen **1**, 243–247 (1950).

The question investigated is: What one-parameter families of transformations have the property that the product of any two transformations of the family still belongs to the family? With certain continuity and monotonicity conditions placed upon the family a definitive answer is obtained involving two arbitrary, continuous and strongly monotone functions of a single variable. *J. L. Vanderslice.*

Golubčikov, A. F. On the structure of automorphisms of complex simple Lie groups. Doklady Akad. Nauk SSSR (N.S.) **77**, 7–9 (1951). (Russian)

The relationship between the multiplicities of the elementary divisors of the matrix X and of the derivation ad X is displayed, where X is an element of a Lie algebra belonging to one of the "four great classes." No proofs are given, and the notation is obscure. *G. D. Mostow.*

Matsushima, Yozô. On a type of subgroups of a compact Lie group. Nagoya Math. J. **2**, 1–15 (1951).

Call a connected closed subgroup H of a compact connected Lie group G an S-subgroup if the coset space G/H is a homology-sphere (over the rationals) of odd dimension. By a theorem of Kudo (to appear) H is then $\sim\!\!\!\sim 0$ in G. Theorem I: G is either simple or essentially the product of a simple group and another group which is either D_1 or C_1. The proof is similar to but simpler than the proof of a related result by Montgomery and Samelson [Ann. of Math. (2) **44**, 454–470 (1943); these Rev. **5**, 60]. By reference to that paper the only S-subgroups of B_n and D_n are the conjugates of the standard subgroup B_{n-1}. The main part of the paper is devoted to the other simple groups. Theorem II: Every S-subgroup of A_n is conjugate to the standard subgroup A_{n-1}, for $n\geqq 8$ (this does not hold for, e.g., $n=2$). That H is of type A_{n-1}, follows easily from Montgomery-Samelson [loc. cit.]. That H is conjugate to the standard A_{n-1}, is proved by an intricate argument on the Lie algebra of A_n. Theorem III: Every S-subgroup of C_n is conjugate to the standard subgroup C_{n-1}. Proof similar to that of theorem II. Theorem IV: F_4, E_6, E_7, E_8 have no S-subgroups. This follows easily from the Poincaré-polynomials.

H. Samelson (Ann Arbor, Mich.).

Koshiba, Zen'ichiro. Remarks on the postulates of metric groups. Osaka Math. J. **3**, 49–53 (1951).

Let G be a group and also a complete metric space and consider the properties: (R) right translations are continuous, (L) left translations are continuous, (D) multiplication is continuous (from both sides at once), (I) inversion is continuous. The author proves that R and L implies D and that D implies I when G is also locally compact. D. Montgomery, whose paper [Bull. Amer. Math. Soc. **42**, 879–882 (1936)] is apparently unknown to the author, proved the former of these and also that D implies I when G is also separable. *R. Arens* (Los Angeles, Calif.).

Montgomery, Deane. Finite dimensional groups. Ann. of Math. (2) **52**, 591–605 (1950).

Let G represent a locally compact topological group which possesses a countable basis of open sets and is of finite dimension. In the present article the author establishes the existence of some open set U containing the identity of G which is, topologically, the product of a compact zero-dimensional set Z and of a connected, locally connected, invariant local group C of same dimension as G. This accomplishes a partial extension of the known result that, in case G is compact or solvable, such an U exists which is the group product of Z and C, Z being in addition also a group and C a local Lie group. *L. Nachbin.*

Tits, J. Sur les groupes triplement transitifs continus; généralisation d'un théorème de Kerékjártó. Compositio Math. **9**, 85–96 (1951).

A triply transitive (i.e., simply transitive on the ordered triples) group of transformations of a topological space E is called continuous if in a certain induced topology it is a topological group. A group is projective if it is the full projective group of a projective line over a commutative field. The author shows that every triply transitive continuous group of the circle or of the 2-sphere is projective, and is then, by Pontryagin's results on topological fields, the projective group of the real, resp. the complex projective line, thus giving a new proof of a theorem of Kérékjártó's. The proofs are short, and employ the author's previous simple axiomatic characterization of projective groups among the triply transitive groups by means of involutions. For the circle continuity of the group can be replaced by the weaker requirement that all transformations be topological. Then a stronger theorem is proved: Every triply transitive continuous group on a manifold (other spaces, e.g., polyhedra, are also admitted) is projective (and so one of the two groups named above; and the manifold is either the circle or the 2-sphere). The proof uses, besides the author's own technique, Freudenthal's theorem on endpoints of groups. *H. Samelson* (Ann Arbor, Mich.).

Vilenkin, N. Ya. Generalized normal divisors of topological groups. Doklady Akad. Nauk SSSR (N.S.) **76**, 625–628 (1951). (Russian)

The author cites Eilenberg and MacLane [Ann. of Math. (2) **43**, 757–831 (1942), p. 761; these Rev. **4**, 88] and Kodaira [Proc. Phys.-Math. Soc. Japan (3) **23**, 67–119 (1941), p. 75; these Rev. **2**, 317] as well as his own experience to indicate the need for an adequate theory of non-closed subgroups of topological groups, and of non-open homomorphisms of groups.

A generalized topological group G is defined in which the operations of product and inverse are continuous as usual

but the topology need not include any separation axiom. A generalized normal divisor (g.n.d.) of G is a system $\mathfrak{N}=[N; U_\alpha]$, comprising a normal divisor N of G, not necessarily closed in G, and a set of neighborhoods U_α such that (1) N is in the intersection of all U_α, (2) for any α and β there is a γ such that $U_\gamma \subset U_\alpha \cap U_\beta$, (3) for any α there is a β such that $U_\beta U_\beta^{-1} \subset U_\alpha$, and (4) for any α and any x in G there is a β such that $x^{-1}U_\beta x \subset U_\alpha$. Now, neighborhoods of the identity of the factor group $G/\mathfrak{N}=G_1$ are defined by the sets NU_α. In particular, to each normal divisor N there is the naturally associated generalised divisor given by $[N; NV_\alpha]$, with V_α traversing a complete set of neighborhoods of e. If f is a continuous map of G upon a G_1, and if the set $[V_\alpha]$ comprises a basis for neighborhoods of the identity e of G_1, then the system $\mathfrak{N}=[f^{-1}(e); f^{-1}(V_\alpha)]$ is a generalized normal subgroup of G.

Since g.n.d.'s are not merely groups, but subgroups in association with a neighborhood system, a special definition is devised of the isomorphism of two g.n.d.'s, when these are found in distinct groups G and G'; the definition involves the direct product of G and G'. Two concluding theorems are as follows. Theorem 5: Let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be g.n.d. in G; then $\mathfrak{N}_1\mathfrak{N}_2/\mathfrak{N}_1 \sim \mathfrak{N}_2/\mathfrak{N}_1 \cap \mathfrak{N}_2$. Theorem 7: Let $\mathfrak{N}_1=[N_1; U_\alpha^1]$ and $\mathfrak{N}_2=[N_2; U_\alpha^2]$ be g.n.d. in G_1, respectively G_2, and suppose $G_1/\mathfrak{N}_1 \sim G_2/\mathfrak{N}_2$; then there is in $G_1 \times G_2$ a g.n.d. $\mathfrak{N}=[N; U_\gamma]$, such that $G_1\mathfrak{N}=G_2\mathfrak{N}=G_1 \times G_2$, and $G_1 \cap \mathfrak{N}=\mathfrak{N}_1$ while $G_2 \cap \mathfrak{N}=\mathfrak{N}_2$. The author remarks that theorem 5 is not true in the traditional theory, due to the fact that $\mathfrak{N}_1$ and $\mathfrak{N}_2$ associated with N_1 and N_2 have an intersection $\mathfrak{N}_1 \cap \mathfrak{N}_2$ which is not the one naturally associated with $N_1 \cap N_2$. *L. Zippin* (Flushing, N. Y.).

Ritt, J. F. Differential groups of order two. Ann. of Math. (2) **53**, 491–519 (1951).

In his paper which introduced the subject of differential groups [Ann. of Math. (2) **51**, 756–765 (1950); these Rev. **11**, 639] the author determined all differential groups of order 1; in his second paper on the subject [ibid. (2) **52**, 708–726 (1950); these Rev. **12**, 241] he gave the theory of structure constants for such groups. In the paper under review he introduces a concept of equivalence of differential groups, and then presents 13 different types of differential groups of order 2 such that every differential group of order 2 is equivalent to a differential group of one of these types. Three of these types yield formal Lie groups after adjunction of a finite number of derivatives, and are said to be of "finite type"; ten of these involve substitutions and therefore do not lead to Lie groups, and are said to be of "substitutional type." [In formula (19) the second equations should read $z_2=u_2+v_2(x+u_2)$.] *E. R. Kolchin.*

NUMBER THEORY

Gupta, Hansraj. A table of values of Liouville's function $L(t)$. Res. Bull. East Panjab Univ. **1950**, 45–63 (1950).

The function $L(x)$ is defined by $L(x)=\sum_{n \le x}\lambda(n)$ where $\lambda(n)$ is the Liouville function whose value is $+1$ or -1 according as the number of prime factors of n, each counted with its multiplicity, is even or odd. The author has condensed an original table of $L(n)$ for $n=1(1)20000$ by giving $L(n)$ for $n=5(5)20000$ and a device for reading out the missing values. The purpose of the table is to study the as yet unproved conjecture of Pólya to the effect that, for $n>1$, $L(n)$ is nonpositive. The conjecture is verified as far as 20000, though there are such examples as $L(19680)=-10$. The table would support the conjecture $L(x)=O(x^{\frac{1}{2}})$ which, like Pólya's conjecture, would imply the Riemann hypothesis. *D. H. Lehmer* (Berkeley, Calif.).

Gloden, A. Factorisation de nombres de 13 à 16 chiffres de la forme X^4+1. Mathesis **55**, 81–82 (1945).

Fifty four complete factorizations of X^4+1 are given for values of X between 1390 and 9893. All but the first two are products of at least three primes. *D. H. Lehmer.*

Lietzmann, W., und Proksch, R. Einige Bemerkungen über die Quersumme natürlicher Zahlen. Math.-Phys. Semesterber. **2**, 135–138 (1951).

Let $Q(x)$ denote the sum of the digits of the natural number x written in the decimal system. Certain simple properties of this function are mentioned. For example, the equation $x+Q(x)=a$ (a being given) is considered. *H. W. Brinkmann* (Swarthmore, Pa.).

Sierpiński, W. Sur les puissances du nombre 2. Ann. Soc. Polon. Math. **23**, 246–251 (1950).

Two theorems are proved concerning terminal and initial digits of 2^n when written to the base 10^k. The first theorem proves that the sequence 1, 2, 4, 8, $\cdots$ when taken modulo 10^k ultimately becomes periodic of proper period $4 \cdot 5^{k-1}$, the first term of the period being 2^k. The second theorem asserts that any number m of k digits being given, there exists infinitely many powers of 2 whose first k digits are respectively the k digits of m. *D. H. Lehmer.*

Sierpiński, W. Sur la périodicité mod m de certaines suites infinies d'entiers. Ann. Soc. Polon. Math. **23**, 252–258 (1950).

Let $u_1, u_2, \cdots$ be an infinite sequence of integers. The author writes $\{u_n\} \varepsilon F_m$ if $\{u_n\}$ is periodic (mod m) (by periodic the author means periodic from a certain point on), $\{u_n\} \varepsilon F$ if $\{u_n\}$ is periodic mod m for every m. The author proves among others the following theorems: (1) If $\{u_n\} \varepsilon F_m$, $\{v_n\} \varepsilon F_m$, then $\{u_n+v_n\} \varepsilon F_m$, $\{u_n \cdot v_n\} \varepsilon F_m$. (2) $\{a^n\} \varepsilon F$ for every a, $n^n \varepsilon F$, $\{n^{n^n}\} \varepsilon F$, etc. (3) If $\{u_n\} \varepsilon F_m$, $v_n \geqq 0$, $\lim v_n=\infty$, $\{v_n\} \varepsilon F$, then $\{u_{v_n}\} \varepsilon F_m$. (4) If $\{u_n\} \varepsilon F_m$, $\{\sum_{k=1}^{n} u_k\} \varepsilon F_m$. These results imply the periodicity of all the usually occurring sequences in number theory. The author remarks that $v_n \geqq 0$, $\{v_n\} \varepsilon F$ does not imply $\{2^{v_n}\} \varepsilon F$; e.g., $v_n=n!([n^{\frac{1}{2}}]-[(n-1)^{\frac{1}{2}}]$ (compare with (3)). *P. Erdös* (Aberdeen).

Somayajulu, B. S. K. R. On Euler's totient function $\phi(n)$. Math. Student **18** (1950), 31–32 (1951).

The author proves that $\limsup \varphi(n+1)/\varphi(n)=\infty$ and $\liminf \varphi(n+1)/\varphi(n)=0$. *P. Erdös* (Aberdeen).

Erdös, Paul. On a conjecture of Klee. Amer. Math. Monthly **58**, 98–101 (1951).

Let $S_k(m)$ denote the number of solutions of $\varphi(x)=m$, where x has exactly k prime factors which appear to the first power. The author proves that $S_k(n!)>cn^k/(\log n)^k$ for every k and sufficiently large n. Earlier, Klee [Amer. Math. Monthly **56**, 21–22 (1949)] had shown that $S_0(n!)>0$, and Gupta [Amer. Math. Monthly **57**, 326–329 (1950); these Rev. **11**, 714] that $S_1(n!) \to \infty$. *R. D. James.*

von Mises, Richard. Über "kleinste" Lösungen diophantischer Gleichungen. Math. Nachr. **4**, 97–105 (1951).

An integral solution $(x_1, \cdots, x_n)$ of the $m<n$ inhomogeneous equations $\sum_{\nu=1}^{n} a_{\mu\nu}x_\nu = b_\mu$ $(1 \leqq \mu \leqq m)$ is a "smallest" solution if there is no distinct solution $(y_1, \cdots, y_n)$ for which $0 \leqq y_\nu \operatorname{sgn} x_\nu \leqq |x_\nu|$ $(\nu = 1, \cdots, n)$. Such solutions have an application in probability theory [cf. the author, Rev. Fac. Sci. Univ. Istanbul (N.S.) **4**, 145–163 (1939)]. It is almost intuitive that only a finite number of smallest solutions exist since the solutions form a $k \leqq n-m$ dimensional inhomogeneous lattice. Explicit bounds for the x_ν are given for $m = 1, 2$ in terms of the $a_{\mu\nu}$, b_μ. *J. W. S. Cassels.*

Pietrosanti, Aldo. Sopra una questione proposta da A. Moessner. Boll. Un. Mat. Ital. (3) **6**, 32–35 (1951).

The author proves that the system of Diophantine equations $A^2+B^2=C^2+D^2$, $A^3+B^3=C^3+D^3$ has no solution in integers other than the trivial one $A=C$, $B=D$.

W. H. Simons (Vancouver, B. C.).

Potts, D. H. The Diophantine equation $x^2+y^2=z^3$. Bull. Calcutta Math. Soc. **42**, 99–100 (1950).

In this paper the author gives an elementary derivation of the complete solutions in rational integers of the Diophantine equation $x^2+y^2=z^3$, $xyz \neq 0$. [Cf. same Bull. **38**, 21–24 (1946); these Rev. **7**, 505.] *W. H. Gage.*

* **Mordell, L. J. Note on cubic equations in three variables with an infinity of integer solutions.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 77–79. Centre National de la Recherche Scientifique, Paris, 1950.

Reprinted from Ann. Mat. Pura Appl. (4) **29**, 301–305 (1949); these Rev. **11**, 714.

Rosser, J. B. Generalized ternary continued fractions. Amer. Math. Monthly **57**, 528–535 (1950).

This paper illustrates the application of the author's algorithm [same Monthly **48**, 662–666 (1941); these Rev. **3**, 161] for solving the linear Diophantine equation to the problem of simultaneous approximation of irrationals involved in the tempering of musical scales treated previously by Barbour [ibid. **55**, 545–555 (1948); these Rev. **10**, 244]. The method consists in taking simple linear combinations of previously obtained approximations in such a way as to minimize the absolute values of the coefficients involved. As a result, the author obtains 9 possible methods of tempering a scale with an "octave" of n notes with $12<n\leqq 118$.

D. H. Lehmer (Berkeley, Calif.).

Brun, Viggo. Music and ternary continued fractions. Norske Vid. Selsk. Forh., Trondheim **23**, 38–40 (1950).

This paper is concerned with a problem discussed by J. M. Barbour [Amer. Math. Monthly **55**, 545–555 (1948); these Rev. **10**, 284] of the division of the octave into equally tempered intervals. The problem is that of simultaneously approximating the irrational ratios $\log \frac{4}{3} : \log \frac{3}{2} : \log 2$ by integers $x:y:z$. The author points out that an algorithm given by him in 1919 gives a simple solution of the problem and gives the results of Barbour (and a little more) for this particular problem. The algorithm is as follows: Given a set of three real numbers a, b, c, each with an associated triplet of integers

$$\begin{array}{ll} a & (\alpha_1, \alpha_2, \alpha_3) \\ b & (\beta_1, \beta_2, \beta_3) \\ c & (\gamma_1, \gamma_2, \gamma_3) \end{array}$$

we replace the row corresponding to the greatest of a, b, c by a row whose number is the difference between the greatest and the midmost of a, b, c and whose triplet is formed by adding the corresponding elements of the triplets belonging to the greatest and midmost of a, b, c. The process is now repeated. Initially, the numbers a, b, c, are given and have triplets $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. Quotations are given from correspondence between Stieltjes and Hermite pointing out defects in the Jacobi ternary continued fraction method of solving the problem. *D. H. Lehmer.*

Rédei, L. Einfacher Beweis des quadratischen Reziprozitätssatzes. Math. Z. **54**, 25–26 (1951).

A proof of the quadratic reciprocity theorem is given which is a variation of Gauss's third proof. It is based upon finding an exact expression for the number of nonnegative solutions of the Diophantine equation $px+qy=k$ in terms of bracket symbols, which seems to be of interest for its own sake. The proof of the theorem comes about through the fact that the same combination of bracket symbols arises when Gauss's lemma is applied in the usual manner.

H. W. Brinkmann (Swarthmore, Pa.).

Jabotinsky, Eri. The minimal Tarry-Escott problem. Riveon Lematematika **4**, 41–58 (1950). (Hebrew. English summary)

The minimum Tarry-Escott equalities

$$\sum_{i=1}^{M} A_i^k = \sum_{i=1}^{h} B_i^k \quad (k = 0, 1, \cdots, M-1),$$

where the A's and B's are integers, are replaced by congruences in which the A's and B's are polynomials in five parameters and the modulus is, in reality, Sylvester's cyclotomic polynomial $\psi_M(z)$ whose roots are $z = 2\cos 2\pi r/M$ with $r \leqq \frac{1}{2}M$ and r prime to M. By choosing as one of the five parameters a root of $\psi_M(z)=0$ and the other four parameters integers, the congruences become equalities and one obtains a minimum solution of the Tarry-Escott problem of any degree $M-1$. Unfortunately, the solution is not in rational integers but in algebraic integers belonging to the real subfield of the cyclotomic field $K(\exp(2\pi i/M))$. Thus, for $M=12$ the solution is in the field $K(\sqrt{3})$. If one permits general algebraic integral solutions the Tarry-Escott problem disappears. *D. H. Lehmer* (Berkeley, Calif.).

Subba Rao, M. V. Congruence properties of $\sigma(n)$. Math. Student **18** (1950), 17–18 (1951).

By the simple expedient of adding conjugate divisors, the author proves a theorem of Ramanathan [Proc. Indian Acad. Sci., Sect. A. **19**, 146–148 (1944); Math. Student **11**, 33–35 (1943); **13**, 30 (1945); these Rev. **6**, 37; **7**, 273] to the effect that the sum of the divisors of $jn-1$ is divisible by j, where j is any divisor of 24 greater than 2, and a theorem of Gupta [Math. Student **13**, 25–29 (1945); these Rev. **7**, 273] stating that the theorem is true for no other values of j.

D. H. Lehmer (Berkeley, Calif.).

Popadić, Milan S. A relation between the prime numbers. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **3**, no. 3, 14 pp. (1950). (Serbo-Croatian. English summary)

The author denotes by $\lambda_n(x)$ the number of integers x which are divisible by no prime greater than the nth prime p_n, and remarks that $x=\lambda_n(x)$ if and only if $x=p_n$. Also the equation $x=\lambda_n(x)+1$ has the unique solution $x=p_{n+1}$. An impractical formula for $\lambda_n(x)$ in the form of a multiple sum

involving the greatest integer function and the logarithms of the first n primes is given and generalized.

D. H. Lehmer (Berkeley, Calif.).

v. Vlaardingen, M. On a formula of Eisenstein. Simon Stevin **28**, 55–59 (1951). (Dutch)

The author gives a short elementary proof of a formula of Eisenstein. From it can be deduced the representation of a prime $4n+1$ as a sum of two squares, and the representation of a prime $3n+1$ in the form x^2-xy+y^2.

W. J. LeVeque (Manchester).

van der Pol, Balth., and Speziali, Pierre. The primes in $k(\rho)$. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 9–15 (1 plate) (1951).

This paper presents, in graphical representation, a list of primes in the quadratic field defined by a primitive cube root of unity. The diagram is based on regular hexagons and gives all primes whose norms do not exceed 10000. The symmetry of the diagram produces a strikingly beautiful effect. There is also a table of c, β in the partition $p=c^2+3\beta^2$ of every prime p of the form $6m+1$ less than 10000.

D. H. Lehmer (Berkeley, Calif.).

Inkeri, K. On the least prime quadratic residue. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **73**, 10 pp. (1950).

Since 2 is a quadratic residue of primes of the form $8n+1$, the problem of an upper bound for the least prime quadratic residue $r=r(p)$ of a prime p is reduced to the two cases $p=8n+3$ and $p=8n+5$. A theorem is given in each of these cases: (I) With the exception of the primes 3, 7, 11, 19, 43, 67, 163, and at most one other prime $>5\cdot 10^9$, $r<(p/3)^{\frac{1}{2}}(1-p^{-\epsilon})$ for $p=8n+3$ and for every $\epsilon>0$. (II) With the exception of a finite number of primes $p=8n+5$, $r<(p/5)^{\frac{1}{2}}$. The proofs depend on estimates for the class number of the field $K((-p)^{\frac{1}{2}})$. Results of Heilbronn and Linfoot and of Siegel are used.

D. H. Lehmer (Berkeley, Calif.).

Knödel, Walter. Ein Satz über Primzahlen. Anz. Öster. Akad. Wiss. Math.-Nat. Kl. **1949**, 112–116 (1949).

Let y and z be positive integers, y being fixed and $y\leqq 2\log x$. $A(z)$ is the number of prime factors of z and $B(z)$ the number of different prime factors of z. Let $l(x, y, E)$ be the number of primes $p\leqq x$ such that y is the least positive integer for which $p-y=z$ for some z satisfying $\min\{A(z)-1, B(z)\}=E$. The author's result is: (A) There exist values of $y\leqq 2\log x$ such that $l(x, y, E)\geqq c_1 x/\log^2 x$ for some constant c_1, and the number $Z_1(y)$ of such y satisfies $Z_1(y)\geqq c_2\log x(\log\log x)^{1-E}$ (the last inequality is presumably under the restriction $E\geqq 1$). This is stated to be a particular case of a result of van der Corput published in the Monatshefte für Mathematik in 1938, but the reviewer has not been able to verify this as he cannot find the paper in that journal for 1938. Possibly the paper intended is that in Math. Ann. 116, 1–50 (1938) in which the representation of numbers y in the form $K\rho+K'\rho'$ is considered. The proof of the second part of (A) depends upon the result (Hilfssatz 3)

$$l(x, y, E)\leqq c_3\frac{x}{\log^2 x}(\log\log x)^E\sum_{q|y}\frac{1}{q},$$

which is stated to be due to Kai-Lai Chung [Sci. Rep. Nat. Tsing Hua Univ. (A) **4**, 249–255 (1940); these Rev. **3**, 68]. The reviewer has been unable to find this result in the paper quoted which contains no mention of a parameter such as E; it is possible, however, that it may be an extension of Chung's main lemma in which there is a similar expression with E replaced by 2. Hilfssätze 1 and 2, as stated by the author, are meaningless, although their intended meaning is obvious from the proofs. The author claims that, by taking z to be a prime p' (i.e., $E=0$) in (A), it follows that: (B) There exists an infinity of different values y_0 each of which is the difference between infinitely many prime pairs p, p'. The reviewer can see no justification for this claim, although (B) is no doubt true. That there exists a y_0 which is the difference between infinitely many prime pairs will follow from (A) if it is shown that there exist values of x arbitrarily large for which the least y for which $l(x, y, E)\geqq c_1 x/\log^2 x$ is uniformly bounded. This the author has not done.

R. A. Rankin (Cambridge, England).

Knödel, Walter. Sätze über Primzahlen. Monatsh. Math. **55**, 62–75 (1951).

The following result, which is a generalisation of the result (A) of the preceding review, is proved: Let k be an integer greater than unity. Then there exist at least $c_1(k)\log^{k-1}x(\log\log\log x)^{-k}$ sets $\{y_1, y_2, \cdots, y_k\}$ of non-negative integers such that $y_1=0$, $y_i<2\log x$ $(1\leqq i\leqq k)$ and with the property that with each such set there are associated at least $c_2(k)x(\log x)^{-k}$ positive integers $z<x$ such that the k numbers $z+y_i$ $(1\leqq i\leqq k)$ are all prime. This is deduced from the theorem that to each set $\{y_1, y_2, \cdots, y_k\}$ there are associated at most $c_3(k; y_1, y_2, \cdots, y_k)x(\log x)^{-k}$ integers $z<x$ for which the numbers $z+y_i$ $(1\leqq i\leqq k)$ are all prime. This latter result is obtained as a straightforward generalisation (from 2 to k) of Brun's method as given in chapter 2, §§2 and 3 of E. Landau's book, "Über einige neuere Fortschritte der additiven Zahlentheorie" [Cambridge Univ. Press, 1937]. The paper commences with a repetition of the result (B) claimed to have been proved in the paper reviewed in the preceding review. There are numerous minor printers' errors.

R. A. Rankin (Cambridge, England).

Monna, A. F. P-adic numbers. Simon Stevin **28**, 40–54 (1951). (Dutch)

An expository talk presented at the University of Leiden.

W. J. LeVeque (Manchester).

Pall, Gordon. Sums of two squares in a quadratic field. Duke Math. J. **18**, 399–409 (1951).

The author uses $r_2(d)$ to denote the number of representations of d as the sum of the squares of two rational integers. He proves: a quadratic form f can be expressed as the sum of squares of two linear forms with integral coefficients if and only if $f=d_1 f_1$, where f_1 is primitive, positive definite or semidefinite, and of a square determinant m^2, and d_1 is a positive integer such that no prime of the form $4n+3$ divides d_1 to an odd exponent. If $m^2\neq 0$, the number of representations is $2r_2(d_1)$; if $m^2=0$, the number is $r_2(d_1)$. These results are applied to find the number of representations of $a+2k\rho$ in the ring $R(\rho)$ of quadratic integers specifically for $\rho^3=-1$, 2 or $\rho^2+\rho+1=0$. Investigation of other cases by students of the author is promised. Reference is made to the allied work of Hel Braun [J. Reine Angew. Math. **178**, 34–64 (1937)] and Mordell [Math. Z. **35**, 1–15 (1932)].

B. W. Jones (Boulder, Colo.).

Lehmer, Emma. The quintic character of 2 and 3. Duke Math. J. 18, 11–18 (1951).

A criterion for the cubic character of 2 with respect to a prime p was given by Gauss in terms of a certain representation of p by a quadratic form. Similar results are given by the author for the quintic character of 2 and 3. The simplest way of stating these results is in terms of the constant term P_5 of the equation whose roots are the quintic periods of the pth roots of unity. In fact, it is proved that 2 is a quintic residue of p if and only if P_5 is even, and 3 is a quintic residue of p if and only if P_5 is multiple of 3. These criteria can also be stated in terms of certain quadratic representations of p.

H. W. Brinkmann (Swarthmore, Pa.).

Cohen, Eckford. Sums of products of polynomials in a Galois field. Duke Math. J. 18, 425–430 (1951).

Extending some previous results [same J. 16, 85–90 (1949); these Rev. 10, 354] the author determines the number of solutions of

$$F=\alpha_1 X_1^{s} Y_1+\cdots+\alpha_\lambda X_\lambda^{s} Y_\lambda + \alpha_{\lambda+1} P_1^{s} X_{\lambda+1}^{s} Y_{\lambda+1}+\cdots+\alpha_\nu P_{\nu-\lambda}^{s} X_\nu^{s} Y_\nu,$$

in primary polynomials $X_1, \cdots, X_l, X_{\lambda+1}, \cdots, X_{\lambda+m}$ of degree k ($l>0$, $m\geqq 0$), $Y_1, \cdots, Y_l$ of degree $ks+t$ and $Y_{\lambda+1}, \cdots, Y_{\lambda+m}$ of degree ks and in arbitrary polynomials $X_{l+1}, \cdots, X_\lambda, X_{\lambda+m}, \cdots, X_\nu$ of degree $<k$, $Y_{l+1}, \cdots, Y_\lambda$ of degree $<ks+t$, and $Y_{\lambda+m+1}, \cdots, Y_\nu$ of degree $<ks$; the P_i are of degree $t/s>k$ and have no nontrivial divisors of degree $<k$. The number of solutions is given in terms of a divisor function $\mathfrak{S}_\nu^{s}(F, k)$ that has appeared previously. The proof makes use of the representation-theory of the reviewer [ibid. 14, 1121–1137 (1947); these Rev. 9, 337].

L. Carlitz (Durham, N. C.).

Hasse, Helmut. Zum Existenzsatz von Grunwald in der Klassenkörpertheorie. J. Reine Angew. Math. 188, 40–64 (1950).

Let Ω be a fixed field of finite degree over the field P of rationals. If K is cyclic over Ω, then, for every fixed finite or infinite prime $\mathfrak{p}$ of Ω, the mapping of the element $\alpha\neq 0$ of the $\mathfrak{p}$-adic completion $\Omega_\mathfrak{p}$ of Ω on the norm residue symbol $(\alpha, \mathrm{K}/\mathfrak{p})$ is a homomorphism $\chi_\mathfrak{p}$ of the multiplicative group $\Omega_\mathfrak{p}^{\times}$ of $\Omega_\mathfrak{p}$ into the Galois group G of K/Ω. Conversely, if S is a finite set of primes of Ω and if, for every $\mathfrak{p}$ in S, a homomorphism $\chi_\mathfrak{p}$ of $\Omega_\mathfrak{p}^{\times}$ into an abstract cyclic group G of order n is given, the question arises whether there exist cyclic fields K of degree n over Ω such that $(\alpha, \mathrm{K}/\mathfrak{p})=\chi_\mathfrak{p}(\alpha)$ for all $\mathfrak{p}$ in S and all $\alpha\neq 0$ in $\Omega_\mathfrak{p}^{\times}$, when G is identified in a suitable manner with the Galois group of K. Grunwald [J. Reine Angew. Math. 169, 103–107 (1933)] asserted that this was always true. However, Shianghaw Wang [Ann. of Math. (2) 49, 1008–1009 (1948); these Rev. 10, 231] gave a simple counterexample. In a subsequent paper [Ann. of Math. (2) 51, 471–484 (1950); these Rev. 11, 489] Wang settled the whole question, by showing that if certain necessary restrictions are imposed, then Grunwald's theorem and certain refinements are correct.

Hasse's paper was written without knowledge of Wang's second paper. He attacks the same problem using Grunwald's original method with the necessary changes and some simplifications. As in Wang's paper, the additional necessary condition is obtained from the product formula for the norm residue symbol. Suppose that K is cyclic of degree 2^ν over Ω with $\nu\geqq 3$, and that the intersection of K with the field of the 2^ν-th roots of unity over P is a real field, say of degree $2^{\nu-\mu}$ over P with $2\leqq\mu<\nu$. Let ζ be a primitive $2^{\nu+1}$-st root of unity and set $\lambda=(\zeta+\zeta^{-1})^2$. Then

$$(*)\qquad \prod_{\mathfrak{p}}\left(\frac{\lambda^{2^{\nu-1}}, \mathrm{K}}{\mathfrak{p}}\right)=1$$

where $\mathfrak{p}$ ranges over the primes of Ω dividing 2 for which λ is not a square in $\Omega_\mathfrak{p}$. In dealing with Grunwald's problem we may assume that the order n of G is a power l^ν of a rational prime number l. For odd l, Grunwald's theorem is true. For $l=2$, (*) yields the necessary condition which must be imposed. Hasse, after adding one prime to the set S in some cases, constructs an infinite set $\mathfrak{F}$ of primes of Ω such that for every $\mathfrak{q}$ in $\mathfrak{F}$ there exists a solution K of the problem for which $\mathfrak{q}$ is completely ramified in K and all other ramified primes belong to S. This set $\mathfrak{F}$ is obtained by means of a certain Abelian field Λ over Ω; it consists of all primes $\mathfrak{q}$ of Ω not in S for which the Frobenius symbol $(\Lambda/\mathfrak{q})$ is a primitive element of a certain cyclic subgroup of the Galois group of Λ/Ω. Using this form of the theorem, Hasse can extend the results to Abelian fields. This question was not treated in Wang's paper.

Finally, Hasse deals with the case that for the primes $\mathfrak{p}$ of S only ramification orders $e_\mathfrak{p}$ and residue class degrees $f_\mathfrak{p}$ are assigned and discusses the necessary and sufficient conditions for the existence of corresponding cyclic fields. As was also pointed out by Wang, the application of Grunwald's theorem to the theory of algebras is not affected by the modification necessary in the theorem.

R. Brauer.

Delange, Hubert. Quelques formules asymptotiques de la théorie des nombres. C. R. Acad. Sci. Paris 232, 1392–1393 (1951).

Several results are stated which may be proved by the author's Tauberian theorems [same vol. 465–467, 589–591 (1951); these Rev. 12, 405, 497], based solely on the facts that the Riemann ζ-function and Dirichlet's L-function do not vanish for $\Re(s)\geqq 1$. Let k be an integer greater than 1 and (l) be a system of integers $l_1, l_2, \cdots, l_r$, all prime to k and not congruent (mod k) to each other. Denote by $E[k, (l)]$ the set of integers n with the property that each prime factor p of n satisfies one of the congruences: $p\equiv l_1, p\equiv l_2, \cdots, p\equiv l_r \pmod k$. Define $\omega(n)$ as the number of different prime factors of n, and $\Omega(n)$ as the total number of prime factors in the decomposition of n. Let $\phi(k)$ denote the number of integers not greater than and prime to k. Landau showed [Handbuch der Lehre von der Verteilung der Primzahlen, Teubner, Leipzig-Berlin, 1909, p. 668] that the ratio of $E[k, (l)]$ to $x[(\log x)^{1-r/\phi(k)}]^{-1}$ tends to a finite positive limit as $x\to\infty$. Related formulas are stated when additional conditions are imposed on $\Omega(n)$ and on the system of integers (l). Other results are also stated such as $\sum_{n\leq x, n\in E}[\omega(n)-\log\log n]^2/\log\log n\sim Ax$ when $x\to\infty$, where A is a positive constant.

S. Ikehara.

Jarník, Vojtěch, et Knichal, Vladimír. Sur le théorème de Minkowski dans la géométrie des nombres. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. 47 (1946), 171–185 (1950).

This is a French version of a paper which appeared earlier in Czech [Rospravy II. Třídy České Akad. 53, no. 43 (1943); these Rev. 8, 565]. For later literature on these questions, see C. A. Rogers [Proc. London Math. Soc. (2) 51, 440–449 (1949); these Rev. 10, 683].

H. Davenport.

Varnavides, P. The Minkowski constants associated with quadratic forms near x^2-2y^2. Bull. Soc. Math. Grèce **25**, 153–163 (1951). (Greek)

Let $d=b^2-4ac>0$ be the discriminant of the quadratic form $f(x, y)=ax^2+bxy+cy^2$. Minkowski showed that if x_0, y_0 are arbitrary real numbers, then there exist integers x, y such that $|f(x+x_0, y+y_0)| \leqq \frac{1}{4}\sqrt{d}$, and the constant $\frac{1}{4}$ cannot be improved since it is attained for $f(x, y)=xy$, $x_0=y_0=\frac{1}{2}$. The author denotes by $M(f)=M(a, b, c)$ the lower bound of all values λ having the property that for arbitrary real x_0, y_0 there exist integers x, y such that $|f(x+x_0, y+y_0)| \leqq \lambda\sqrt{d}$ and calls $M(f)$ the constant of Minkowski associated with the form f. Heinhold [Math. Z. **44**, 659–686 (1930)] and Davenport [Nederl. Akad. Wetensch., Proc. **49**, 815–821; these Rev. **8**, 444] obtained $M(f_0)=1/4\sqrt{2}$ for the form $f_0(x, y)=x^2-2y^2$. In the present paper the author asserts that for certain forms f differing little from $f_0(x, y)$ the Minkowski constant $M(f)<1/4\sqrt{2}-10^{-7}$ and states that this result is a consequence of the following theorem: Let g, h be arbitrary real numbers satisfying $|g-\sqrt{2}|<\tau^{-14}$, $|h-\sqrt{2}|<\tau^{-14}$, where $\tau=1+\sqrt{2}$, and let $\xi=x+gy$, $\eta=x-hy$. Then, given real numbers x_0, y_0 there exist integers x, y such that $|(\xi-x_0)(\eta-y_0)| \leqq \frac{1}{2}-\tau^{-14}$ unless $g=h=\sqrt{2}$ and $x_0=u+(v+\frac{1}{2})\sqrt{2}$, $y_0=u-(v+\frac{1}{2})\sqrt{2}$, u and v being integers. Since g and h are nearly $\sqrt{2}$, the form $f(x, y)=(x+gy)(x-hy)$ is "near" x^2-2y^2. The proof is practically unreadable because of the large number of misprints, of which we note the following: p. 154, line 13, x^2-2y^2 instead of x^2-2y; p. 158, equation (29), $\xi-x$ instead of $E-x$; the sign $<$ should be replaced by $>$ in (30), (33), (36), (39), (44), and p. 159, line 15. The relations (46), (47), (50), (53), and (54) are so badly printed that they are meaningless.

T. M. Apostol (Pasadena, Calif.).

Davis, C. S. The minimum of a binary quartic form. I. Acta Math. **84**, 263–298 (1951).

Every binary real quartic form without real roots can be mapped into a uniquely determined form $x^4+6mx^2y^2+y^4$, $-\frac{1}{3}<m \leqq \frac{1}{3}$, by an appropriate affine transformation. In the present paper the critical lattices of the region of points (x, y) for which $x^4+6mx^2y^2+y^4 \leqq 1$ (i.e., lattices of minimum determinant with no points other than the origin interior to the region) are determined. For $-\frac{1}{3}<m<\frac{1}{3}$ there are two critical lattices, the one generated by the vectors (α, α), (β, δ), where (α, α), (β, δ) and $(\alpha+\beta, \alpha+\delta)$ are on the boundary of the region and the reflection of this lattice by the y-axis. The determination of these lattices is reduced to a minimum problem which is solved with the use of the invariants of the quartic form. For $-\frac{1}{3}<m \leqq -\frac{1}{5}$ there are two lattices generated by P, Q such that P, Q, $P+Q$, $P-Q$ are on the boundary of the region and each lattice is obtained from the other by an affine mapping of the region into itself. By use of the method developed by Mordell [Proc. London Math. Soc. (2) **48**, 339–390 (1945); these Rev. **6**, 257] for the solution of the corresponding problem for the cubic forms the author shows these two lattices are the only critical lattices of the region. Let $f(x, y)$ be a real quartic form without real roots, $\mathfrak{D}$ its discriminant, and m the uniquely defined number such that the region of points (x, y) for which $f(x, y) \leqq 1$ is mapped into the region defined by $x^4+6mx^2y^2+y^4 \leqq 1$ by an affinity. The ultimate result of the paper is the construction of the best possible function $k(m)$ so that, for all forms $f(x, y)$ with given m and $\mathfrak{D}$, integers x, y exist, not both of which are zero, so that $f(x, y) \leqq k(m)\mathfrak{D}^{\frac{1}{4}}$. This result is equivalent to the determination of the determinant of the critical lattices of the region of points x, y for which $x^4+6mx^2y^2+y^4 \leqq 1$. *D. Derry.*

Chalk, J. H. H., and Rogers, C. A. On the product of three homogeneous linear forms. Proc. Cambridge Philos. Soc. **47**, 251–259 (1951).

Let Λ be a lattice of points $X=(x_1, x_2, x_3)$ of determinant Δ. Let $P(X)=|x_1x_2x_3|$. Davenport [Proc. London Math. Soc. (2) **44**, 412–431 (1938)] showed that there is a point A of Λ not the origin such that $P(A) \leqq \Delta/7$. This result is improved upon by showing that Λ has an infinity of sets of generators A, B, C for which $P(A)P(B)P(C) \leqq (\Delta/7)^3$ and, except for a stated case, $P(A)P(B)P(C) \leqq (\Delta/7.2)^3$.

L. Tornheim (Ann Arbor, Mich.).

Clarke, L. E. On the product of three non-homogeneous linear forms. Proc. Cambridge Philos. Soc. **47**, 260–265 (1951).

Let θ, ϕ, ψ be the roots of t^3-4t+2. Let $\xi=u+v\theta+w\theta^2$, $\eta=u+v\phi+w\phi^2$, $\zeta=u+v\psi+w\psi^2$. Then corresponding to any real numbers a, b, c there exist rational integers u, v, w for which $|(\xi-a)(\eta-b)(\zeta-c)| \leqq \frac{1}{7}$, and for certain values a, b, c, always $|(\xi-a)(\eta-b)(\zeta-c)| \geqq \frac{1}{7}$ for all rational integers u, v, w. This result for the cubic field $R(\theta)$ is analogous to theorems of Davenport [Proc. Cambridge Philos. Soc. **43**, 137–152 (1947); these Rev. **8**, 444] for the cyclic cubic fields of discriminant 49 and 81 and of Prasad [Nederl. Akad. Wetensch., Proc. **52**, 240–250, 338–350 = Indagationes Math. **11**, 55–65, 112–124 (1949); these Rev. **11**, 12] for discriminant -23. The choice here is the totally real non-cyclic field with smallest discriminant, viz. 148. The result implies that $R(\theta)$ has a Euclidean algorithm.

L. Tornheim (Ann Arbor, Mich.).

ApSimon, H. On the critical lattices of the 'quadrifoil.' Quart. J. Math., Oxford Ser. (2) **2**, 17–25 (1951).

Let R_a be the set of all points (x, y) of the plane interior to at least one of the circles $(x \pm a)^2+y^2=1$, $x^2+(y \pm a)^2=1$, $0<a \leqq 1$. The above paper determines the critical lattices of the regions R_a. For $1/\sqrt{5}<a \leqq 1$ there is only one such lattice and this is generated by a boundary point of R_a which is on the line $x+y=0$ and the reflection of this point about the x-axis. For $0<a<1/\sqrt{5}$ each critical lattice contains one of the above points P and two boundary points Q, T of R_a with $T=Q+P$, while for $a=1/\sqrt{5}$ lattices of both types exist. The proofs make use of the fact that critical lattices of subregions of R_a which are admissible for R_a are likewise critical for R_a. *D. Derry* (Vancouver, B. C.).

Hlawka, Edmund. Bemerkungen zu einem Satz von R. Rado. Anz. Öster. Akad. Wiss. Math.-Nat. Kl. **1950**, 219–226 (1950).

Let $f(x)$ be a bounded nonnegative function vanishing outside a bounded set and integrable in the Riemann sense over n-dimensional space R_n. Suppose that there is a non-singular matrix A so that $f(A(x-y)) \geqq \min\{f(x), f(y)\}$ for all x, y. Let Γ be any lattice with determinant 1. A theorem of R. Rado [J. London Math. Soc. **21**, 34–47 (1946); these Rev. **8**, 444] asserts that

$$(*) \qquad f(o)+\tfrac{1}{2}\sum{}'f(g) \geqq |A|\,V, \quad \text{where} \quad V=\int f(x)\,dx,$$

the sum being taken over all the points g of Γ other than the origin o and the integral being over R_n. The author discusses this theorem and its improvement by J. W. S. Cassels [J. London Math. Soc. **22**, 196–200 (1948); these

Rev. 10, 19]. He shows that if equality holds in (*) then every point of R_n can be represented in the form $x+A^{-1}g$, where g is in Γ and x is in the closure of the set of points y with $f(y)>0$. The author remarks that by use of one of his earlier theorems [Math. Z. 49, 285–312 (1943); these Rev. 5, 201] it follows that if f, A, and $\epsilon>0$ are given it is possible to find a lattice Γ with determinant 1 so that

$$f(o)+\tfrac{1}{2}\sum' f(g)\leqq\tfrac{1}{2}V+f(o)+\epsilon.$$

Let f_1 and f_2 be two functions satisfying the same conditions as f (with the same matrix A) and let Γ be a lattice with determinant 1. Then, if $|A|(V_1+V_2)^2>V_1\sum_1+V_2\sum_2$ where $\sum_i=f_i(o)+\frac{1}{2}\sum' f_i(g)$ and $V_i=\int f_i(x)dx$, every point of R_n is of the form $\pm(y_1-y_2)+A^{-1}g$, where g is in Γ and $f_i(A^{-1}y_i)>0$, $i=1, 2$.

A p-adic form of Rado's theorem is given.

C. A. Rogers (London).

Gel'fond, A. O., and Fel'dman, N. I. On the measure of relative transcendentality of certain numbers. Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 493–500 (1950). (Russian)

The authors improve an earlier result due to Gel'fond [Doklady Akad. Nauk SSSR (N.S.) **64**, 277–280 (1949); these Rev. **10**, 682]. Let α be a root of an irreducible equation of degree 3 and $\alpha\neq 0, 1$ an algebraic number. Let $P(x, y)$ be any polynomial in x and y having integral coefficients each of modulus not exceeding H and of degrees n_1 and n_2 in x and y. Then, for any $\epsilon>0$,

$$|P(\alpha^a, \alpha^{a^2})|\geqq e^{-\sigma^{4+\epsilon}}$$

provided that $\sigma=\max(n_1+n_2, \log H)>\sigma_0$. The proof makes use of ideas similar to those introduced in the paper referred to above.

R. A. Rankin (Cambridge, England).

Cassels, J. W. S. Some metrical theorems in Diophantine approximation. V. On a conjecture of Mahler. Proc. Cambridge Philos. Soc. **47**, 18–21 (1951).

Denote by $\|\xi\|$ the difference between ξ and the nearest integer. It is well known that if $\theta_1, \theta_2, \cdots, \theta_n$ are any n real numbers then (1) $\max_{j=1,2,\cdots,n}\|g\theta_j\|\leqq g^{-1/n}$ has infinitely many integer solutions $g>0$. In particular, (1) holds if $\theta_j=\alpha^j$. Khintchine showed that, if $g^{-1/n}$ is replaced by $\epsilon g^{-1/n}$, (1) has infinitely many solutions for almost all $\theta_1, \theta_2, \cdots, \theta_n$; he also showed that for almost all α, $\max_{j=1,2,\cdots,n}\|g\alpha^j\|<\epsilon g^{-1/n}$ has infinitely many solutions. On the other hand, it is well known that for any $\epsilon>0$ and almost all $\theta_1, \cdots, \theta_n$

$$\max_{j=1,2,\cdots,n}\|g\theta_j\|<g^{-1/n-\epsilon}$$

has only a finite number of solutions. Mahler conjectured [Math. Ann. **106**, 131–139 (1932)] that for any ϵ and almost all α, $\max_{j=1,2,\cdots,n}\|g\alpha^j\|<g^{-1/n-\epsilon}$ has only a finite number of solutions. For $n=1$ this is well known. Recently Kubilyus proved it for $n=2$ [Doklady Akad. Nauk SSSR (N.S.) **67**, 783–786 (1949); these Rev. **11**, 82]. For $n=2$ the author proves the following more general theorem: Let $\phi(g)$ and $\psi(g)$ satisfy $\sum_g\phi(g)\psi(g)<\infty$, $\psi(g)\geqq\max\{\phi(g), g^{-1}\log g\cdot d(g)\}$, where $d(g)$ is the number of divisors of g. Then for almost all α there are only a finite number of solutions of

$$\|g\alpha\|\leqq\phi(g), \qquad \|g\alpha^2\|\leqq\psi(g).$$

Mahler's conjecture follows by putting $\psi(g)=g^{-1+\epsilon}$, $\phi(g)=g^{-1-\epsilon}$. Finally the author states the following conjecture which would imply Mahler's conjecture for general n. For all $\epsilon>0$ there is some $\delta>0$ depending only on ϵ and such that $\delta\to 0$ as $\epsilon\to 0$ with the following property: The number of solutions N of

$$0\leqq r<g, \qquad \|r^j/g\|\leqq r^{1-1/n-\epsilon} \qquad (j=2, \cdots, n)$$

satisfies $N=O(g^{1/n+\delta})$.

P. Erdős (Aberdeen).

***Sanders, Johannes Marinus. Verdelingsproblemen bij gegeneraliseerde duale breuken. [Distribution Problems for Generalized Binary Fractions].** Thesis, Free University of Amsterdam, 1950. 88 pp.

If the interval $[0, 1)$ is divided into two subintervals $\Delta_{1,0}$ and $\Delta_{1,1}$ and each of these is subdivided into two more, giving $\Delta_{2,0}$, $\Delta_{2,1}$, $\Delta_{2,2}$, $\Delta_{2,3}$, etc., the sequence π of subdivisions $V_0, V_1, \cdots$ so formed is called a distribution process. Let the points of V_n be $0=c_{n,0}<c_{n,1}<\cdots<c_{n,2^n}=1$, so that $\Delta_{n,r}=[c_{n,r}, c_{n,r+1})$, and let the length of $\Delta_{n,r}$ be $d_{n,r}$. Any number $\theta\in[0, 1)$ lies in just one interval Δ_{n,r_n} for each n, and if we put $c_n=0$ or 1 according as r_n is even or odd, we have a mapping $\theta\to\pi$, $c_1c_2\cdots$ which reduces to the ordinary binary expansion $\theta=0\cdot c_1c_2\cdots$ when $c_{n,r}=r\cdot 2^{-n}$ for $0\leqq r\leqq 2^n$, $n\geqq 0$. This thesis extends many of the known results concerning binary expansions to general processes. A process π is said to be normal when the above correspondence is one-to-one; it is shown that normality is equivalent to the relation $\lim_{n\to\infty}(\text{norm } V_n)=0$. Let $N(n, x)$, for $x\in[0, 1]$, be the number of points $c_{n,r}$ of V_n for which $0\leqq c_{n,r}\leqq x$. Then $\lim_{n\to\infty}N(n, x)\cdot 2^{-n-1}=\Phi(x)$ is called the distribution function of π, and it is shown that Φ is continuous and non-decreasing in $[0, 1]$, with $\Phi(0)=0$, $\Phi(1)=1$. In going from V_n to V_{n+1}, the interval $\Delta_{n,r}$ is divided into the two intervals $\Delta_{n+1,2r}$ and $\Delta_{n+1,2r+1}$. Write the subratios $d_{n+1,2r}/d_{n+1,2r+1}$ as $\lambda_{n,r}/\mu_{n,r}$, where $\lambda_{n,r}+\mu_{n,r}=1$, and let $V(x, \delta)$ be the set of numbers $\lambda_{n,r}$ corresponding to those subintervals $\Delta_{n,r}$ which lie entirely in the interval $[x-\delta, x+\delta]$. Then $\underline{\lambda}(x)=\lim_{\delta\to 0}(\liminf V(x, \delta))$ and $\bar{\lambda}(x)=\lim_{\delta\to 0}(\limsup V(x, \delta))$ are the lower and upper first ratio functions, respectively. A normal process π is said to be regular if $\underline{\lambda}(x)=\bar{\lambda}(x)$ for $x\in[0, 1]$. This common first ratio function $\lambda(x)$ is continuous in $[0, 1]$, and has the value $\frac{1}{2}$ at any point of differentiability of $\Phi(x)$. If $\lambda(x)$ is constant, π is called quasiuniform, and π is termed uniform if the subratios themselves are all equal.

Some of the principal results are these: (1) If P_n is the number of occurrences of a given combination of m digits in the first $N+m-1$ digits of the sequence π, $c_1c_2\cdots$ corresponding to θ, and p and q are the numbers of zeros and ones in this combination, then $\lim_{N\to\infty}P_N/N=(\lambda(\theta))^p(1-\lambda(\theta))^q$ for almost all θ if π is regular. This generalizes Borel's theorem on normal numbers. (2) Let $\theta_n\to\pi$, $c_{n+1}c_{n+2}\cdots$ for $n=1, 2, \cdots$. Then (a) the regularity of π is sufficient for the sequence $\{\theta_n\}$ to possess an asymptotic distribution function for almost all θ; (b) if π is regular, then this distribution function is independent of θ for almost all θ if and only if π is quasiuniform; (c) if π is regular, $\{\theta_n\}$ is uniformly distributed (mod 1) for almost all θ if and only if π is uniform. (3) In the case of a uniform process, estimates are given for the quantities $R_{\alpha,\beta}(N)$ and $R^*(N)$ associated with the sequence $\{\theta_n\}$ (for notation, see Koksma [Diophantische Approximationen, Springer, Berlin, 1936]). In particular, it is shown that for fixed α, β, $R_{\alpha,\beta}(N)=O(N^{\frac{1}{2}}\log^{\frac{1}{2}(3+\epsilon)}N)$ for every $\epsilon>0$ and almost all θ, and the same bound obtains for $R^*(N)$ for almost all θ. (4) Two questions asked by C. J. Everett [Bull. Amer. Math. Soc. **52**, 861–869 (1946); these Rev. **8**, 259], concerning real number expansions somewhat similar to those of the present paper, are answered.

W. J. LeVeque (Manchester).

ANALYSIS

Gustin, W. S. Asymptotic behavior of mean value functions. Amer. Math. Monthly **57**, 541–544 (1950).

The author examines the limits of $M(t)=(\sum_{\nu=1}^{n}a_\nu x_\nu^t)^{1/t}$, $\sum_{\nu=1}^{n}a_\nu=1$, for $t=0, +\infty, -\infty$ with the aid of the concepts of weak and strong asymptotic equivalence. Two functions f and g are weakly, resp. strongly, asymptotic to each other ($f\sim g$, resp. $f\approx g$) as $t\to t_0$ if they are both weakly, resp. strongly, asymptotic to the same formal power series $F=\sum_{k=0}^{\infty}c_k x^k$, that is if $f(t)=\sum_{k=0}^{k}c_k(t-t_0)^k+o(t-t_0)^k$ (for finite t_0) or if $f(t)=\sum_{k=0}^{k}c_k/t^k+O(t^{-k})$ (for $t_0=\pm\infty$) with $k=0, 1, 2, \cdots$, resp. if the same relation holds also between any kth derivative $f^{(k)}$ of $f(t)$ and the kth formal derivative $F^{(k)}$ of F.

By his method the author gains, besides the results to be obtained by the usual method [see, e.g., Hardy, Littlewood, and Pólya, Inequalities, Cambridge, 1934, §§2, 3], such as, e.g., $a_n^{1/t}x_n<M(t)<x_n$ for $t>0$, $\lim_{t\to\infty}M(t)=x_n$, where $x_n=\max_\nu x_\nu$, also the remarkable theorem asserting that $M^{(k)}(t)$ and $[\log M(t)]^{(k)}$ are both positive near $t=-\infty$ and near $t=+\infty$ if k is odd, and are both negative near $t=\pm\infty$ if k is even. For $k=2$ this gives the convexity of $M(t)$ and $\log M(t)$ near $t=-\infty$ and the concavity of both functions near $t=+\infty$, which contains a result of H. Shniad [Bull. Amer. Math. Soc. **54**, 770–776 (1948); these Rev. **10**, 238]. *J. Aczél* (Miskolc).

Thielman, H. P. On a pair of functional equations. Amer. Math. Monthly **57**, 544–547 (1950).

The author solves the pair of functional equations (A) $f(xy)=g(x)^{p(y)}h(y)^{q(x)}$, (B) $r(xy)=p(x)q(y)$ by reducing (A) to the form (B); the equation (B) has the general solution $p(t)=c_1t^b$, $q(t)=c_2t^b$, $r(t)=c_1c_2t^b$ [also satisfying $r(xy)=p(y)q(x)$] which can be easily obtained from the general solution $F(x)=x^b$ of Cauchy's functional equation $F(xy)=F(x)F(y)$. [Reviewer's remark: the author's way of reducing (A) to (B) is unnecessarily long, as the substitution $f(t)=F(t)^{r(t)}$, $g(t)=G(t)^{q(t)}$, $h(t)=H(t)^{p(t)}$ immediately gives the equation $F(xy)=G(x)H(y)$ of the form (B).] Thus we get $g(t)=(d_1x^a)^{c_2x^b}$, $h(t)=(d_2x^a)^{c_1x^b}$, $f(t)=(d_1d_2x^a)^{c_1c_2x^b}$. The author also solves the pair of functional equations $F(xy)=p(y)G(x)+g(x)H(y)$, $r(xy)=p(x)q(y)$ by reducing them to the system (A), (B). *J. Aczél* (Miskolc).

Sard, Arthur. Remainders: functions of several variables. Acta Math. **84**, 319–346 (1951).

In a previous paper [Duke Math. J. **15**, 333–345 (1948); these Rev. **10**, 197] the author interpreted the linear functionals on $C^n(0, 1)$ and $C^{n-1}(0, 1)$ which annul polynomials of degree $n-1$ as remainders in approximation. The present paper treats the case when the interval $0\leqq t\leqq 1$ is replaced by the hypercube $0\leqq t_i\leqq 1$, $i=1, \cdots, m$. *D. G. Bourgin.*

Dahlgren, Lars. A theorem on translations by Hille, and its interpretation from the point of view of the theory of probability. Skand. Aktuarietidskr. **33**, 184–192 (1950).

Let $f(x)$ be continuous and bounded in $0\leqq x<\infty$ and let $0\leqq a<\infty$. According to Hille the function $\exp(xh^{-1}\Delta_h)\cdot f(a)$ tends to $f(x)$ as $h\to 0$, and this uniformly in each finite (x, a)-interval. This statement is easily seen to be equivalent to

$$e^{-a/h}\sum\frac{1}{k!}\left(\frac{x}{h}\right)^k f(a+kh)\to f(x).$$

The left side is related to the Poisson distribution in the same way as the S. Bernstein interpolation formula is to the Bernoulli distribution. The two convergence theorems can be proved in a simple way basing the necessary estimates on the Chebyshev inequality. *W. Feller.*

Morozov, M. I. On certain questions of the uniform approximation of continuous functions by functions from interpolation classes. Doklady Akad. Nauk SSSR (N.S.) **77**, 381–383 (1951). (Russian)

A Chebyshev system $\{\varphi_k(x)\}_{k=1}^n$ of real-valued continuous functions on $[0, 1]$ is one having the property that a non-trivial linear combination (1) $a_1\varphi_1(x)+\cdots+a_n\varphi_n(x)$ has at most n roots on $[0, 1]$. It is well known that such a system has the following extremal properties [cf. e.g. S. Bernstein, Leçons sur les propriétés extrémales . . . , Gauthier-Villars, Paris, 1926, pp. 1–5]: If $f(x)$ is a given continuous function, (a) for a set of $n+1$ distinct points E_{n+1}, there is a unique function (1) deviating least from $f(x)$ on E_{n+1}; (b) there is a unique function (1) deviating least from $f(x)$ on $[0, 1]$ and the deviation is the least upper bound of the deviations for all sets E_{n+1}; (c) the extremal function in (b) is characterized by the fact that its difference from $f(x)$ attains its maximum absolute value at at least $n+1$ points with alternating signs. The author announces that he has extended these results to approximating functions much more general than (1), under much weaker hypotheses than have been used in previous work in this direction. He uses a single-valued function $F(x; a_1, \cdots, a_n)$, $0\leqq x\leqq 1$, $a_i'\leqq a_i\leqq a_i''$, continuous as a function of its $n+1$ arguments, and denotes by $D(F_x)$ the set of all values taken by F for fixed x and arbitrary a_i. The function F is to have the following "interpolation property": every system

$$y_i=F(x_i; a_1, \cdots, a_n), \qquad i=1, \cdots, n,$$

where $\{x_k\}$ are all different and $y_i\in D(F_{x_i})$, has a unique solution $(a_1, \cdots, a_n)$. The function F determines the class $K(F)$ of continuous $f(x)$ each of which arises by assigning a set of parameters $\{a_k\}$; this is the "interpolation class" of the title. Then (a), (b), (c) remain true if the class of linear combinations (1) is replaced by $K(F)$. *R. P. Boas, Jr.*

Stuloff, N. Die Differentiation beliebiger reeler Ordnung. Math. Ann. **122**, 400–410 (1951).

The author defines a fractional derivative as the limit of a fractional difference quotient and discusses some of its properties. Let $f(x)$ be defined for $x>x_0$; then $f^{[\alpha]}(x)=\lim_{h\to 0+}h^{-\alpha}\{\Delta^\alpha f(x), h\}$, where $\{\Delta^\alpha f(x), h\}$ is the αth difference of $f(x)$ with span h, if the limit exists. For positive integral α, $f^{[\alpha]}(x)=(-1)^\alpha f^{(\alpha)}(x)$ if the latter derivative exists. The following results are obtained for a nonnegative decreasing $f(x)$. If $f^{[\alpha]}(x)$ exists for $x>0$, $-1\leqq\alpha<0$, then $f^{[\alpha]}(x)$ has the expected integral representation

$$\Gamma(-\alpha)f^{[\alpha]}(x)=\int_0^\infty t^{-\alpha-1}f(x+t)dt;$$

conversely, the existence of this integral implies the existence of $f^{[\gamma]}(x)$ for $-1\leqq\alpha\leqq\gamma<0$. If $xf(x)$ is bounded, $f^{[\alpha]}(x)$ exists for $-1<\alpha<0$. A completely monotonic $f(x)$ on $(0, \infty)$ possesses the mean value theorem

$$h^{-\alpha}\{\Delta^\alpha f(x), h\}=f^{[\alpha]}(x+\alpha\theta h), \qquad 0<\theta<1,$$

for all $\alpha>0$. [For other results in the direction of this

paper, see P. W. Marke, Copenhagen thesis, 1942; these Rev. 8, 66.] *R. P. Boas, Jr.* (Evanston, Ill.).

Tsuji, Matsugu. On positive definite sequences and functions. Tohoku Math. J. (2) 2, 142–165 (1950).

The author extends the theory of positive definite functions from one to several variables and from functions to sequences. But all this has been done long ago. [For functions see the reviewer, Math. Ann. 108, 378–410 (1933); for sequences see the reviewer, S.-B. Preuss. Akad. Wiss. 1933, 371–376, especially p. 376; and for a survey of theory on groups see H. Cartan and R. Godement, Ann. Sci. École Norm. Sup. (3) 64, 79–99 (1947); these Rev. 9, 326]. *S. Bochner* (Princeton, N. J.).

Mambriani, Antonio. Su le funzioni wronskiane. Rivista Mat. Univ. Parma 1, 475–487 (1950).

A function $f(x)$ of class C^1 defined over an interval (a, b) is called a Wronskian function if there exists a function $\varphi(x)$, also of class C^1 and defined over (a, b), such that the determinant $f(x)\varphi'(x)-f'(x)\varphi(x)$ has no zeros in (a, b). The function $\varphi(x)$ is also a Wronskian function, and the functions $f(x)$, $\varphi(x)$ are said to be conjugate. A nontrivial solution of a differential equation $y''+p(x)y'+q(x)y=0$, where $p(x)$, $q(x)$ are of class C^1 in (a, b), is a Wronskian function which is special in that it is of class C^2.

In the first part of this paper the author determines various classes of Wronskian functions. Examples: (1) If $f(x)$, $g(x)$ are Wronskian functions over (a, b), and $f(x)$ has no zero in (a, b), $f(x)g(x)$ is also a Wronskian function. (2) If $f(x)$, $g(x)$ are conjugate Wronskian functions, $c_1f(x)+c_2g(x)$, where c_1, c_2 are any constants (not both 0), is a Wronskian function. In the second part of the paper the author discusses some properties of Wronskian functions and of pairs of conjugate Wronskian functions. Most of the properties are closely related to the properties of solutions of linear differential equations of the second order which are treated in the Sturm-Liouville theory. Examples: (1) The zeros of a Wronskian function are of the first order. (2) Between any two consecutive zeros of a Wronskian function there lies just one zero of any conjugate Wronskian function. (3) If $f(x)$, $g(x)$ are conjugate Wronskian functions, any Wronskian function of the form $c_1f(x)+c_2g(x)$ has a conjugate of the same form. The significance of the paper lies in the fact that this body of properties, which is familiar in its connections with differential equation theory, is given a more general setting. *L. A. MacColl.*

Calculus

***Haupt, Otto, Aumann, Georg, und Pauc, Christian. Differential- und Integralrechnung unter besonderer Berücksichtigung neuerer Ergebnisse. Band II. Differentialrechnung.** 2d ed. Walter de Gruyter & Co., Berlin, 1950. 210 pp. 16.50 DM.

This is the second volume, devoted to differential calculus, of a revised edition of the Haupt-Aumann text of 1938. The first volume, with the subtitle Einführung in die reele Analysis, has already appeared [these Rev. 12, 84; the subtitle was omitted in this review]. The third volume of the revision, on integral calculus, is due to appear later.

Some topics included are: continuity and differentiability, tangents, formal rules of differentiation, derivatives of the elementary functions, higher derivatives, the mean value theorem, Taylor's formula with remainder, Taylor's series, indeterminate forms, existence and uniqueness of primitive functions, elementary integration formulas, divided differences, Newton and Lagrange interpolation polynomials, derivatives (generalized derivatives) derivatives of higher order as limits of divided differences, convex and convexoid functions of nth order, osculating circle and curvature, limit sets of functions of one or more real variables, derivate sets and contingents, nowhere differentiable functions, differentiability properties of rectifiable arcs, contingents and generalized differentials of functions of several variables, differentials of higher order, partial derivatives of first and higher order, Taylor's formula with remainder for functions of several variables, implicit functions, functional determinants, maxima and minima of functions of several variables, multiplier rule, differentiable mappings, functional dependence and independence. There is a short appendix on Lebesgue measure. *O. Frink.*

***Kašanin, Radivoje. Viša matematika. II. Knjiga druga. [Higher Mathematics. II. Part 2].** Naučna Knjiga, Belgrade, 1950. vii+679 pp.

This is the second volume of the second part of a textbook on advanced calculus [the first part was published in 1949; cf. these Rev. 12, 164]. It covers integration, multiple integrals, vector fields, and ordinary differential equations. *W. Feller* (Princeton, N. J.).

***Rimini, Cesare. Fondamenti di analisi matematica con applicazioni.** Nicola Zanichelli, Bologna. vol. 1, xx+1–549 (1948); vol. 2, xv+551–1253 (1951). 4000+6000 lire.

A rather comprehensive textbook designed to provide for physicists, chemists, engineers, etc. a knowledge of such parts of analysis as are most likely to prove useful to them. The chapter headings are as follows. 1. Complex numbers. 2. Determinants, linear equations, linear and quadratic forms. 3. Sets, functions. 4. Limits of sequences and functions. 5. Continuous functions. 6. Infinitesimals and infinities. 7. Derivatives and differentials of functions of one and several variables. 8. Applications in analysis. 9. Series, the elementary transcendental functions in the complex plane. 10. Vectors. 11. Applications to geometry and cinematics. 12. Review of algebraic analysis. 13. Integrals in one and many dimensions. 14. Reduction of many-dimensional integrals, multiple integrals, exact differentials. 15. Calculation of primitive functions. 16. Differential equations. 17. Geometrical applications, Fourier series. 18. Elements of the calculus of variations. 19. Approximate calculations. An appendix on analytic geometry and 230 pages of remarks and exercises close the work.

Bruwier, L. Sur quelques conséquences du théorème de Rolle. Mathesis 59, 225–232 (1950).

The theorem of Rolle is applied to the determinant with elements $a_{ij}=f_j(a_i)$, $j=1, \cdots, n-1$; $a_{in}=1$, $i=1, \cdots, n$; and $a_{nj}=f_j(x)$, $j=1, \cdots, n-1$, where the $f_j(x)$ have finite $(n-1)$-st derivatives in a closed interval containing the a_i. Variations are obtained by specializing the $f_j(x)$. *P. Civin* (Eugene, Ore.).

Ulčar, Jože. Über die Vorzeichen der partiellen Ableitungen einer Funktion von zwei Veränderlichen. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire 3, no. 2, 8 pp. (1950). (Macedonian. German summary)

Concerns the intersection of a surface with the tangent plane. *W. Feller* (Princeton, N. J.).

Tricomi, F. G. On the theorem of Frullani. Amer. Math. Monthly 58, 158–164 (1951).

With suitable restrictions on $f(x)$ Frullani's formula is

$$(1) \qquad \int_0^\infty [f(ax)-f(bx)]x^{-1}dx = f(0)\log b/a.$$

Also there is a more general result,

$$(2) \qquad \int_0^\infty [f(ax)-f(bx)]x^{-1}dx = [f(0)-f(\infty)]\log b/a.$$

The paper under review generalizes these results in two directions. If $f(x)$ is continuous, and $p_1(x)$, $p_2(x)$ are positive and continuously differentiable for $x>0$, then

$$\int_0^\infty \{f[p_1(x)]p_1'(x)/p_1(x) - f[p_2(x)]p_2'(x)/p_2'(x)\}dx$$

$$= L_0 - L_\infty, \text{ where}$$

$$L_0 = \lim_{\delta\to 0}\int_{p_1(\delta)}^{p_2(\delta)} f(y)y^{-1}dy, \qquad L_\infty = \lim_{\omega\to\infty}\int_{p_1(\omega)}^{p_2(\omega)} f(y)y^{-1}dy,$$

whenever these limits exist. If $f(x)$ is an integrable periodic function with period p, if

$$\lim_{x\to 0+} p_2(x)/p_1(x) = \lambda_0, \qquad \lim_{x\to\infty} p_2(x)/p_1(x) = \lambda_1$$

exist, and $M = [\text{integral of } f \text{ over } (0, p)]/p$, then

$$\int_0^\infty \{f[p_1(x)]p_1'(x)/p_1(x) - f[p_2(x)]p_2'(x)/p_2(x)\}dx$$

$$= f(0)\log\lambda_0 - M\log\lambda_1.$$

These results are extended to the complex field for the case where $p_1(x)=ax$, $p_2(x)=bx$. If $x=e^t$, $a=e^\alpha$, $b=e^\beta$, $f(e^t)=F(\xi)$, equation (2) can be written

$$\int_{-\infty}^{\infty} [F(\xi+\alpha)-F(\xi+\beta)]d\xi$$

$$= (\beta-\alpha)[F(-\infty)-F(\infty)]+2\pi iS,$$

where $F(-\infty)$ and $F(\infty)$ are the limit values of $F(\xi)$ as $\xi\to\mp\infty$, and α and β are two arbitrary constants and where there are only a finite number of singular points of $F(\xi)$ on the strip $\Im\alpha\leqq\Im\xi\leqq\Im\beta$ and $F(\xi)\to F(\mp\infty)$ as $\Re\xi\to\mp\infty$, uniformly in the strip (without considering the point at ∞). If F is an analytic function of ξ this form of Frullani's theorem can easily be extended to the complex plane to give

$$\int_0^\infty [f(ax)-f(bx)]x^{-1}dx = (\log b-\log a)[f(0)-f(\infty)+2\pi iS],$$

where (i) the imaginary parts of both logarithms lie in $(-\pi, \pi)$; (ii) if γ is the sector $\arg a\leqq\arg x\leqq\arg b$, then S represents the sum of the residues of $f(x)$ inside γ, each divided by the affix of the corresponding singular point; $f(0)$ and $f(\infty)$ represent respectively the limit values of $f(x)$ as $|x|\to 0$ and $|x|\to\infty$ in γ with the condition that $f(x)$ approaches these limits uniformly with respect to $\arg x$.

R. L. Jeffery (Kingston, Ont.).

Maravall Casesnoves, Darío. Rigorous mathematical theory of the singular functions of quantum mechanics. Revista Mat. Hisp.-Amer. (4) 10, 238–245 (1950). (Spanish)

The purpose of this paper is to give a mathematically rigorous theory of the Dirac delta function and other singular functions used in quantum mechanics. The basic idea is the natural one of interpreting the equation $f(0)=\int_{-\infty}^{\infty} f(x)\delta(x)dx$ as a conventional symbolic equivalent of an equation of the form $f(0)=\lim_{\alpha\to\infty}\int_{-\infty}^{\infty} f(x)g(x,\alpha)dx$. The results obtained in the course of the development of this idea do not go beyond what are already matters of common knowledge.

L. A. MacColl (New York, N. Y.).

Milne-Thomson, L. M. Tensor calculus by direct methods. Revista Mat. Hisp.-Amer. (4) 10, 171–195 (1950). (Spanish)

The author, in a notation of his own, defines the dyadic and scalar multiplication of vectors, and defines $A^{(r)}$ as a tensor of rank r if it is a linear operator such that for all $r\geqq 1$ and every vector $\bar{x}$ the scalar products $A^{(r)}\cdot\bar{x}$ and $\bar{x}\cdot A^{(r)}$ are tensors of rank $r-1$. Introducing base vectors, he attempts to show how an affine connection and a metrical space can be obtained. *D. J. Struik.*

Tola Pasquel, Jose. Algebraic theory of motors and their applications in mechanics. Revista Ci., Lima 52, nos. 1–2, 5–48 (1950). (Spanish)

A motor $\mathbf{M}$ is defined as the set $(\mathbf{f}, \Phi, P)$, where $\mathbf{f}$ is the vector and Φ is the moment of the motor with respect to the point P. The motor $(\mathbf{f}_1, \Phi_1, P_1)$ is equal to $\mathbf{M}$ if $\mathbf{f}_1=\mathbf{f}$, $\Phi_1=\Phi$. The author uses dual numbers $x+\epsilon y$, (x, y real, $\epsilon^2=0$) to derive algebraic properties of motors and applies the theory to systems of forces and rigid body movements. A similar discussion with a different notation is given by L. Brand [Tensor and Vector Analysis, Wiley, New York, 1947; these Rev. 9, 68]. *L. M. Milne-Thomson.*

Theory of Sets, Theory of Functions of Real Variables

***Aleksandrov, P. S., and Kolmogorov, A. N. Vvedenie v teoriyu množestv i teoriyu funkcil. Čast' pervaya. [Introduction to the Theory of Sets and the Theory of Functions. Part One].=Aleksandrov, P. S. Vvedenie v obščuyu teoriyu množestv i funkcil. [Introduction to the General Theory of Sets and Functions].** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1948. 411 pp.

This volume is the first of a two-volume treatise dealing with the foundations of latter-day mathematical analysis, designed for students of mathematics in Soviet universities and pedagogical institutes. The thesis is advanced that the broadest notions of set, topological space, continuity, and integral should find a place in the curriculum for all students of higher mathematics; that the study of these concepts only for the line and for n-space is, in view of the development of mathematics in the past half-century, an anachronism. Part one (written by Aleksandrov) deals with topics relating generally to cardinal number and continuity. Part two (written by Kolmogorov) is to treat the theory of integration and its applications to probability, functional analysis, dynamical systems, and the like. The volume under review furnishes powerful arguments in favor of the authors' thesis. It requires very little in the way of previous training

on the reader's part, covers the essential features of the classical theory of real functions, introduces the reader to a large number of modern concepts, and is written in a highly lucid, indeed, elegant style. All of the abstract concepts presented are illuminated by well-chosen examples. An outline of the topics handled will indicate the plan of the work. Chapter I: basic notions about sets; mappings; countable sets; ordered sets; cardinal equivalence of sets; the Schroeder-Bernstein theorem. Chapter II: definition of the real number system in terms of Dedekind cuts in the rational number system; elementary properties of sets of real numbers. Chapter III: basic properties of well-ordered sets; the axiom of choice; the well-ordering theorem (Zermelo's third proof is given); standard theorems on and classification of infinite cardinal numbers. Chapter IV: elementary topology of the line and plane (all usual concepts are introduced and all standard theorems are proved). Chapter V: continuous real-valued functions on the line; functions of bounded variation; the Weierstrass approximation theorem for closed intervals (Bernstein's proof is given; this is one of the few proofs in the book not admitting generalization to the most general context); derivatives. Chapter VI: definition of metric spaces; open and closed sets; Borel sets; subspaces; dense and nowhere dense sets; connectedness; countable bases; continuous mappings of metric spaces; general topological spaces; separation axioms; Urysohn's imbedding theorem. Chapter VII: compact and complete metric spaces; bicompact topological spaces. In conclusion, one may observe that a book of this caliber, if generally accessible to the mathematical world, might well have a deep and beneficial influence on higher mathematical education. *E. Hewitt.*

***Kurepa, Đuro. Teorija skupova.** [The Theory of Sets]. Školska Knjiga, Zagreb, 1951. xix+443 pp.

A textbook on set theory starting from the fundamentals. In chapter 1 (32 pp.) the fundamental concepts and elementary operations are introduced. Chapter 2 (40 pp.) treats cardinal numbers. Chapter 3 (190 pp.) treats partially ordered sets and ordinal numbers. There follow a chapter (60 pp.) on topological and metric spaces and a final chapter (80 pp.) on limiting processes in analysis, measure theory, Borel and Souslin sets. *W. Feller* (Princeton, N. J.).

Kurepa, Georges. Ensembles partiellement ordonnés et ensembles partiellement bien ordonnés. Acad. Serbe Sci. Publ. Inst. Math. **3**, 119–125 (1950).

It is shown that every partially ordered set E contains a subset W with the following two properties. (i) For each $a \in E$ there exists $x \in W$ with $a \leqq x$. (ii) Every simply ordered subset of W is well ordered. *B. Jónsson.*

Sierpiński, Wacław. Le dernier théorème de Fermat pour les nombres ordinaux. Fund. Math. **37**, 201–205 (1950).

The author proves various theorems on ordinal numbers. He proves among others that there are arbitrarily large transfinite ordinals α, β, γ, $\alpha<\beta<\gamma$ so that for all $n=1, 2, \cdots$, $\alpha^n+\beta^n=\gamma^n$. He also remarks that $\omega+10$ is the smallest even transfinite ordinal which is not the sum of two prime ordinals, and that ω^2 is the smallest ordinal not the sum of a finite number of primes. *P. Erdös* (Aberdeen).

Sierpiński, Wacław. Sur un type ordinal dénombrable qui a une infinité indénombrable de diviseurs gauches. Fund. Math. **37**, 206–208 (1950).

The author proves in a simple and ingenious way that there exists a countable order type which has $2^{\aleph_0}$ distinct left divisors. The question of the existence of these order types was raised by Mostowski. *P. Erdös* (Aberdeen).

Simonsen, W. Sur un problème de la théorie des correspondances multivoques entre des ensembles abstraits. Acta Math. **84**, 225–229 (1951).

If P and Q are sets let Corex (P, Q) be the class of many-valued "functions" with domain P and range exhausting Q. The problem proposed is this: For f in Corex (P, Q) and g in Corex (P, R) when is there an h in Corex (Q, R) such that $hf=g$? A necessary and sufficient condition when f is single-valued is that $gf^{-1}f=g$, and then $h=gf^{-1}$. Supposing these conditions fulfilled, the relations of the classes of f, g, and h are studied, where a class of f, for example, is a subset M minimal with respect to the property of containing $f^{-1}fM$ [cf. the author, Acta Math. **81**, 291–297 (1949); these Rev. **11**, 506]. *R. Arens* (Los Angeles, Calif.).

Simonsen, W. Sur la résolution d'un problème de la théorie des correspondances multivoques abstraites. Acta Math. **84**, 301–318 (1951).

The author presents several necessary and sufficient conditions that there should be an h such that $hf=g$ [see the preceding review]. One such condition is: for x in P and z in $g(x)$ there is a y in $f(x)$ such that $f^{-1}(y)$ is contained in $g^{-1}(z)$. By considering the structure of the partially ordered set H of solutions h, which has a greatest element $\bar{h}$ when not void, the author discovers conditions that $\bar{h}$ should be the only solution. Such conditions are (1) f is single-valued, (2) g is single-valued. The proofs are all quite simple and devoid of transfinite methods. *R. Arens.*

Gottschalk, W. H. Choice functions and Tychonoff's theorem. Proc. Amer. Math. Soc. **2**, 172 (1951).

A short proof of a combinatorial lemma of R. Rado [Canadian J. Math. **1**, 337–343 (1949); these Rev. **11**, 238] is based on the theorem that the Cartesian product of compact spaces is compact. A corollary is: A family of finite sets has a one-to-one choice function if and only if each of its finite subfamilies has a one-to-one choice function [cf. C. J. Everett and G. Whaples, Amer. J. Math. **71**, 287–293 (1949); these Rev. **10**, 517]. *L. W. Cohen.*

Fodor, G. On a theorem in the theory of binary relations. Compositio Math. **8**, 250 (1951).

Sharpening a theorem of Lázár the author proves the following theorem. Let M be a set of positive measure in Euclidean space. To every element x of M there corresponds a subset $\varphi(x)$ of M so that x is not contained in $\varphi(x)$ or its closure. Two points x and y are called independent if $x \notin \varphi(y)$, $y \notin \varphi(x)$. A set is called independent if any two of its elements are independent. The author proves that there exists an independent set M' of positive outer measure. *P. Erdös* (Aberdeen).

Mayrhofer, Karl. Über den Zusammenhang der additiven, Inhalts- und Masstheorien. Anz. Öster. Akad. Wiss. Math.-Nat. Kl. **1949**, 141–147 (1949).

Mayrhofer, Karl. Über den Zusammenhang der additiven Inhalts- und Masstheorien. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **158**, 1–36 (1950).

The author continues his previous investigation [Monatsh. Math. **52**, 217–229 (1948); these Rev. **10**, 107], now considering content as well as measure, and developing his theory in a systematic manner. A nonnegative additive set function i defined in a field κ of sets, with $i(0)=0$, is

called an (additive) "content" ("Inhalt") by the author, and the sets of κ are called "i-measurable." In the particular case where κ is a σ-field and i is totally additive, i is called an (additive) "measure." Two other set functions, associated with the content i and defined for the system λ of all the subsets M of each set $A\varepsilon\kappa$, $\bar{i}(M)=\inf i(\bar{A})$, $\underline{i}(M)=\sup i(\underline{A})$ $(M\subset\bar{A}, \underline{A}\subset M, \bar{A}\varepsilon\kappa, \underline{A}\varepsilon\kappa)$ are named by the author "outer function" ("Aussenfunktion") and "inner function" ("Innenfunktion") of i, respectively. On the other hand, a monotone increasing set function $i^*(M)$, defined in a field λ, with $i^*(0)=0$ and $i^*(M_1+M_2)\leqq i^*(M_1)+i^*(M_2)$ is called an "outer content." Every $\bar{i}$ is an outer content, but not conversely. A special case of i^* is the "outer measure" μ^* for which λ is a σ-field and the last inequality is replaced by $\mu^*(\sum_\nu M_\nu)\leqq\sum_\nu\mu^*(M_\nu)$ for every sequence $M_\nu\varepsilon\lambda$. Moreover, the author considers an "inner content," i.e. a monotone increasing set function $i_*(M)$, defined in a field λ, with $i_*(0)=0$ and $i_*(M_1+M_2)\geqq i_*(M_1)+i_*(M_2)$ if M_1 and M_2 are disjoint. Every $\underline{i}$ is an inner content, but not conversely. The content i is "complete" if every set, enclosable in two i-measurable sets of arbitrarily small i-difference, is i-measurable also. The content i is said to have the "cut property" if to every set $M\varepsilon\lambda-\kappa$ there is a set $A\varepsilon\kappa$ with finite i such that $AM\varepsilon\lambda-\kappa$. And i is said to have the [weaker] "part property" if to every set $M\varepsilon\lambda-\kappa$ there is a set $A\varepsilon\kappa$ such that $AM\varepsilon\lambda-\kappa$ and both $\underline{i}(AM)$ and $\underline{i}(A-M)$ are finite. There are complete measures without the cut property and complete contents without the part property.

Carathéodory's measurability condition for a set $M\varepsilon\lambda$ is used either in the form (1) $\bar{i}(L)=\bar{i}(LM)+\bar{i}(L-M)$ or (2) $\underline{i}(L)=\underline{i}(LM)+\underline{i}(L-M)$. For the i-measurability of M $(\varepsilon\lambda)$ it is necessary that (1) and (2) are satisfied for every $L\varepsilon\lambda$. But (1) or (2) (for every $L\varepsilon\lambda$) is sufficient for the i-measurability of M if and only if i is complete and, moreover, i has the cut property in the case of (1) and the part property in the case of (2). For an outer content i^* (or an inner content i_*) the measurable sets M are defined by Carathéodory's measurability condition and so the "content i corresponding to i^* (or i_*, respectively) is obtained. The author calls i^* (or i_*) an "ordinary outer (or inner) content" if i^* is the outer function $\bar{i}$ (or i_* is the inner function $\underline{i}$) of the content i which corresponds to i^* (or i_*). [Here the author uses "ordinary" instead of "regular" (in the sense of Carathéodory).] Every outer function is also an ordinary outer content. Moreover, the contents corresponding at all i^* coincide with the complete contents having the cut property. Hence if for a given content i one forms the outer function $\bar{i}$ and then the content corresponding to $\bar{i}$, one obtains, as an extension of i, a complete content with the cut property. This extension is called the "V-process" by the author. In general by this process one does not obtain the smallest complete content over i. But if one considers a measure m with the property that every m-measurable set is the sum of countably many m-measurable sets with finite m, then the V-process furnishes the smallest complete measure over m. The author then discusses the extension of a totally additive content to a complete measure [see H. Hahn, Ann. Scuola Norm. Super. Pisa (2) 2, 429–452 (1933); H. Hahn and A. Rosenthal, Set Functions, Univ. New Mexico Press, Albuquerque, N. M., 1948, pp. 74–80; these Rev. 9, 504]. If to an ordinary outer content i^* the corresponding content i and subsequently its inner function $\underline{i}$ is formed, then $\underline{i}$ is an ordinary inner content, which is called the "inner content corresponding to i^*". An ordinary inner content i_* is the inner content corresponding to an ordinary outer content i^* if and only if the content i corresponding to i_* has the cut property; in this case $i^*=\bar{i}$. In this theorem i^* can be replaced by an ordinary outer measure μ^* provided that i is a measure. The author then again obtains the reviewer's [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. 1916, 305–321] characterization of the Carathéodory inner measure. *A. Rosenthal* (Lafayette, Ind.).

Mayrhofer, Karl. Über die Homomorphieen einer Struktur. Anz. Öster. Akad. Wiss. Math.-Nat. Kl. 1949, 293–297 (1949).

Let $\mathfrak{S}$ and $\mathfrak{S}'$ be two σ-fields of somas, consider a single-valued mapping of $\mathfrak{S}$ into $\mathfrak{S}'$, and designate by X' the image of X. According to Carathéodory [Ann. Scuola Norm. Super. Pisa (2) 8, 105–130 (1939), p. 109] this mapping is called a homomorphy if for every sequence of somas $\{A_\nu\}$ of $\mathfrak{S}$ one has $(\sum_\nu A_\nu)'=\sum_\nu A_\nu'$ and if, moreover, for each pair (A, B) of $\mathfrak{S}$ one has $(AB)'=A'B'$. The author proves that such a homomorphy can be characterized by the following three properties: (α) $A\subset B$ implies $A'\subset B'$. (β) A being disjoint from B implies that A' is disjoint from B'. (γ) Let $\mathfrak{D}$ be the class of the somas of $\mathfrak{S}$ which are mapped on the empty soma O' of $\mathfrak{S}'$; then for every sequence of somas $\{U_\nu\}$ of $\mathfrak{D}$ one has $(\sum_\nu U_\nu)'=O'$. *A. Rosenthal.*

Aumann, Georg. Ein Beweis des Loomisschen Darstellungssatzes für σ-Somenringe. Arch. Math. 2, 321–324 (1950).

A proof (using Stone's representation theorem) that every Boolean σ-algebra is isomorphic to the quotient algebra of a Boolean σ-algebra of sets modulo a suitable σ-ideal. Since the author does not use the topological aspects of the Stone theory, his proof contains an inelegant and unnecessary use of transfinite induction. *P. R. Halmos* (Montevideo).

Sikorski, R. The integral in a Boolean algebra. Colloquium Math. 2, 20–26 (1949).

Let A be an abstract Boolean σ-algebra and let F be a σ-homomorphism of the algebra B of Borel sets of real numbers into A. The author remarks that F can be considered a generalized point function, and that if A is σ-isomorphic to a quotient algebra X/I, where X is a σ-algebra of subsets of a space S, then there exist real-valued functions on S corresponding to F in the obvious way. If a measure is given on A it can be transferred to X and the integral of F can be defined in terms of corresponding functions on S. *L. H. Loomis* (Cambridge, Mass.).

Sierpiński, Wacław. L'équivalence par décomposition finie et la mesure extérieure des ensembles. Fund. Math. 37, 209–212 (1950).

Given a real number $\mu>0$ and a bounded subset E of Euclidean m-space R_m $(m\geqq 1)$ with m-dimensional outer measure $m^*(E)>0$, when can one assert that E is equivalent by finite decomposition to some set $H\subset R_m$ with $m^*(H)=\mu$? Assuming $2^{\aleph_0}=\aleph_1$, the author shows that E can always be decomposed into two sets each of which has outer measure equal to $m^*(E)$, and from this lemma he deduces that the assertion in question is true whenever $\mu\geqq m^*(E)$, and also when $0<\mu<m^*(E)$ and $m\geqq 3$. On the other hand, in case $m=1$ or 2 and $0<\mu<m^*(E)$, the assertion is false whenever E is measurable. As the author remarks, the above lemma (and its consequences) can be proved without assuming $2^{\aleph_0}=\aleph_1$ by following an idea of Lusin [C. R. Acad. Sci. Paris 198, 1671–1674 (1934), p. 1673], but the proof is then much more difficult. *J. C. Oxtoby* (Bryn Mawr, Pa.).

Hadwiger, H. Ueber die Jordansche Messbarkeit von Vereinigung und Durchschnitt beliebig vieler Punktmengen. Compositio Math. **9**, 80–84 (1951).

For a bounded point-set X in k-dimensional Euclidean space let $J(x)$ and $\underline{J}(x)$ be the outer and inner Jordan measures (content) respectively. Let P_σ be the k-dimensional cube whose centre is P and whose side-length is σ (all these cubes have the same orientation). Write

$$\underline{\alpha}(A)=\inf\{\underline{J}(P_\sigma A)\sigma^{-k}\} \qquad (0<\sigma<1;\ P\varepsilon A),$$
$$\bar{\alpha}(A)=\sup\{\overline{J}(P_\sigma A)\sigma^{-k}\} \qquad (0<\sigma<1;\ P\bar{\varepsilon}A).$$

The author proves two theorems: (1) The sum of any number of uniformly bounded sets A for which $\alpha(A)\geqq\lambda>0$, λ independent of A, is J-measurable. (2) The intersection of any number of uniformly bounded sets B for which $\bar{\alpha}(B)\leqq\mu<1$, μ independent of B, is J-measurable.

H. G. Eggleston (Swansea).

Denjoy, Arnaud. Une extension du théorème de Vitali. Amer. J. Math. **73**, 314–356 (1951).

This paper is a detailed exposition of a series of notes published in C. R. Acad. Sci. Paris **231**, 560–562, 600–601, 737–740, 1013–1015 (1950); these Rev. **12**, 246, 324, 398.

T. H. Hildebrandt (Ann Arbor, Mich.).

Pauc, Christian. La dérivation dans les réseaux incomplets et les fonctions de Haar. C. R. Acad. Sci. Paris **232**, 1387–1389 (1951).

This note covers three points. It indicates a relation between the general Vitali theorem basis of Denjoy [see the preceding review] and the basis of derivation of de Possel [J. Math. Pures Appl. (9) **15**, 391–409 (1936)] used in a recent note of the author [same C. R. **231**, 1406–1408 (1950); these Rev. **12**, 487]. To every set I of the basis $\mathfrak{B}$ of derivation is attached a set $E(I)$ containing I, a sequence of sets M_n contracting on x if $E(M_n)$ contains x and $\lim_n \mu(M_n)=0$. The set $\Omega(\gamma)$ of Denjoy then corresponds to the sum of the sets $E(I')$ intersecting I or γ. In the second place the author points out by an example that if the fundamental family is too meager the strong Vitali theorem in the form, for every $\epsilon>0$ and every subset X of the space, and every subfamily $\mathfrak{B}$ of $\mathfrak{B}$ covering X in the Vitali sense (almost all points of x belong to some contracting sequence of $\mathfrak{B}$) there exists a disjoint sequence $I_1, \cdots, I_n, \cdots$ covering X up to a set of zero measure such that $\mu(\sum I_n)<\text{ext meas } X+\epsilon$. For the example the base used is on the square Q of side unity in the plane, with the network consisting of rectangles of altitude 1 and bases $(p/2^n, (p+1)/2^n)$, the measure function being Lebesgue measure. It is shown that the measure induced by these rectangles is weaker than the Lebesgue measure, and asserted that while a completely additive set function on Q has finite derivative almost everywhere relative to this base, this derivative does not agree with the Radon-Nikodým integrand relative to Lebesgue measure. Finally, if one considers the set of half open intervals $(p/2^n, (p+1)/2^n)$ as a basis for derivation then derivatives of $\int\varphi(t)dt$ give the mean values on these intervals. This is related to the nth partial sum of the expansion of a summable function in terms of the Haar orthogonal system on $(0, 1)$.

T. H. Hildebrandt (Ann Arbor, Mich.).

Halperin, Israel. Non-measurable sets and the equation $f(x+y)=f(x)+f(y)$. Proc. Amer. Math. Soc. **2**, 221–224 (1951).

In connection with the results of Sierpinski and Lusin [C. R. Acad. Sci. Paris **165**, 422–424 (1917)], Hahn and Rosenthal [Set Functions, Univ. of New Mexico Press, 1948, pp. 102–104; these Rev. **9**, 504], Ostrowski [Jber. Deutsch. Math. Verein. **38**, 54–62 (1929)], and Jacobsthal and Knopp [S.-B. Berlin Math. Ges. **14**, 121–129 (1915)], the author introduces the concept of s-nonmeasurable subsets (with respect to the set B); these have zero inner measure and their complements relative to B also have a zero inner measure. He shows that if B is the set of all real numbers, resp. any half-open finite interval, then for every infinite cardinal not greater than the power of the continuum B can be partitioned into k disjoint subsets which are congruent with each other under (ordinary, resp. modulo the length of B) translations. He proves that the sets $E_n=E(c_n)$ of the countable distinct values $c_1, c_2, \cdots, c_n, \cdots$ assumed by a discontinuous solution $f(x)$ of $f(x+y)=f(x)+f(y)$ are a partition of the set of all real numbers into disjoint s-nonmeasurable subsets congruent under translation. The similar theorem for open intervals $a\leqq x<b$ holds if the additional condition $f(b-a)=0$ is satisfied.

J. Aczél (Miskolc).

Kalmár, László. On Cauchy's convergence test. Acta Math. Acad. Sci. Hungar. **1**, 109–112 (1950). (English. Russian summary)

The author gives two new proofs for the Cauchy convergence criterion. The first proof does not assume the Weierstrass-Bolzano theorem. It proceeds by successive subdivision of intervals into overlapping parts, namely the right and left two-thirds, a device which the author attributes to Brouwer. The second proof is obtained by showing that the Cauchy condition is equivalent to the property that any two subsequences differ by a sequence approaching zero. A method of proving the Weierstrass-Bolzano theorem by the selection of a monotonic subsequence is also described. The method used is attributed to the late J. Kürschák.

O. Frink (State College, Pa.).

Kaluza, Th., jr. Gleichmässige Konvergenz von Funktionsgefügen als notwendige und hinreichende Bedingung für die Stetigkeit der Grenzfunktion. Arch. Math. **2**, 334–336 (1950).

Given a sequence of functions $F_n(x)$ on a closed interval $[a, b]$, an arbitrary partition $a=x_0<x_1<\cdots<x_k=b$, and a set of natural numbers $n_1, \cdots, n_k$, the function $g(x)=f_{n_i}(x)$ on $x_{i-1}\leqq x<x_i$, $g(b)=f_{n_k}(b)$ is called a span function for the sequence. The smallest n_i is called the order m of the span function $g(x)$. It is shown that: If $f_n(x)$ is continuous on $[a, b]$ and $\lim_n f_n(x)=f(x)$, then $f(x)$ is continuous on $[a, b]$ if and only if there is a uniformly convergent sequence of span functions $g_n(x)$ for the sequence $f_n(x)$ whose orders $m_n\uparrow\infty$. An example shows that the condition on the m_n is necessary. The theorem leads to a very short proof of the fact that monotone convergence of a sequence of continuous functions to a continuous limit is uniform.

L. W. Cohen (Flushing, N. Y.).

Cantele, Maddalena. Sulle funzioni quasi continue. Boll. Un. Mat. Ital. (3) **6**, 30–32 (1951).

According to L. Tonelli, a function $f(x)$ is called quasi-continuous in an interval I if to every $\epsilon>0$ there exists an open set O with $m(O)<\epsilon$ such that $f(x)$ is continuous in $I-O$. The author gives a very simple example of a quasi-continuous function $f(x)$ which is not continuous in any set $I-E$ with $m(E)=0$: let $f(x)$ be the characteristic function of a linear open set of measure $\frac{1}{2}$, which is dense in the unit interval.

A. Rosenthal (Lafayette, Ind.).

Schmidt, Robert. Zusammenhänge zwischen Differenzenquotienten und Derivierten von stetigen Funktionen. J. Reine Angew. Math. **188**, 112–114 (1950).

In E. Hopf's dissertation [Berlin, 1926], there is the following theorem: If the real functional $f(x)$ is continuous in the neighbourhood of $x=a$, if $D(f;x)$ is one of the derived numbers of $f(x)$, if the linear combination $D(f;x)+\lambda[ax](f)$, $\lambda>0$, tends to a limit L as $x\to a$ where $[ax](f)$ is the difference quotient at a, then every derivative of $f(x)$ is continuous at $x=a$. Furthermore, $f(x)$ is differentiable at $x=a$ and $f'(x)=L/(1+\lambda)$. This and other considerations led the author to prove: If $f(x)$ is continuous, bounded in the open, closed, or half open interval I, if $D(f,x)$ is one of the derived numbers of $f(x)$, if $x_0\varepsilon I$, $\rho>0$, $\sigma\geqq 0$, $\rho+\sigma=1$, if $F(x)=\rho D(f;x)+\sigma[xx_0](f)$, then, for all $x\varepsilon I-x_0$,

$$\liminf_{x\to x_0} F(x)\leqq\liminf_{x\to x_0}[xx_0](f),$$

$$\limsup_{x\to x_0}[xx_0](f)\leqq\limsup_{x\to x_0} F(x).$$

R. L. Jeffery (Kingston, Ont.).

Kuiper, N. H. Distribution modulo 1 of some continuous functions. Nederl. Akad. Wetensch., Proc. **53**, 1390–1396 = Indagationes Math. **12**, 460–466 (1950).

Let $f(x)$ be increasing for sufficiently large x. Assume that $\lim_{x\to\infty} f(x)/ax^{a-1}=k>0$, where $a>0$. Let φ be an unbounded set and let $\varphi(x)$ denote the Lebesgue measure of the intersection of φ with the interval $(0,x)$. The author defines $V(\varphi)=\lim_{x\to\infty}\varphi(x)/x$ if this limit exists. He proves that $V(\varphi)=V[f^{-1}(\varphi)]$. He applies this result to the theory of C-distributions investigated by the author and Meulenbeld [same Proc. **52**, 1151–1157, 1158–1163; these Rev. **11**, 423]. One of his results is the following: Let $\lim_{x\to\infty} f(x)=\infty$, $f(x)$ continuous; if, whenever $\gamma>\beta>M$ and $f(\gamma)-f(\beta)>\frac{1}{2}$ then $\beta(f(\gamma)-f(\beta))/(\gamma-\beta)<L$ (M and L positive constants), then $f(x)$ does not possess a C^1 distribution mod 1.

P. Erdös (Aberdeen).

Bononcini, Vittorio E. Sulle funzioni lipschitziane di due variabili. Rivista Mat. Univ. Parma **1**, 449–457 (1950).

Let $0<\alpha<1$ and $1<\beta\leqq 2$. The author constructs (i) a function of two real variables x, y on the unit square $0\leqq x\leqq 1$, $0\leqq y\leqq 1$, that is ACT (absolutely continuous in Tonelli's sense) [see S. Saks, Theory of the Integral, Warsaw-Lvov, 1937, p. 169] and does not belong to Lip α, but has first order partial derivatives belonging to L^β, (ii) another function of x, y that is ACT and in Lip α, but whose first order derivatives are not in L^β. *H. P. Mulholland.*

Pucci, Carlo. Derivazione per serie di funzioni a variazione limitata. Boll. Un. Mat. Ital. (3) **5**, 281–286 (1950).

The author's first theorem is a rephrasing of Fubini's theorem on term-by-term differentiation of series of monotone functions. The second theorem gives another sufficient condition for term-by-term differentiation, in terms of length of arc. *L. M. Graves* (Chicago, Ill.).

Pucci, Carlo. Serie di funzioni a variazione limitata. Boll. Un. Mat. Ital. (3) **6**, 1–3 (1951).

A remark on term-by-term differentiation of series of functions of bounded variation. *L. M. Graves.*

Kober, H. On decompositions and transformations of functions of bounded variation. Ann. of Math. (2) **53**, 565–580 (1951).

A function $\phi(t)$, $0\leqq t\leqq 1$, is a co-variation (COV) to $k(t)$, $0\leqq t\leqq 1$, if ϕ is nondecreasing and if, given $\epsilon>0$, there is a $\delta=\delta(\epsilon)>0$ such that, for any sequence of points $0\leqq t_1<T_1<t_2<T_2<\cdots<t_n<T_n\leqq 1$ at each of which ϕ is continuous and for which $\sum\{\phi(T_j)-\phi(t_j)\}<\delta$, it is true that $\sum V_{t_j,T_j}k<\epsilon$, where $V_{t,T}k$ is the variation of k on the interval $\langle t, T\rangle$. The functions $g(t)$, $0\leqq t\leqq 1$, and $h(t)$, $0\leqq t\leqq 1$, are contra-variations (CAV) if they are the positive and negative variations, respectively, of $f(t)=g(t)-h(t)$, so that f is of bounded variation, and $V_{0,t}f=V_{0,t}g+V_{0,t}h=g(t)+h(t)$, $0\leqq t\leqq 1$. The functions g and h are normal contra-variations (NCAV) if they are CAV and at each point t in $(0,1)$ the value of $f(t)$ lies between $f(t-)$ and $f(t+)$. Necessary and sufficient conditions in order that two functions be CAV, or NCAV, are established. If $f(t)$, $0\leqq t\leqq 1$, is of bounded variation, the following generalization of the Lebesgue decomposition theorem is demonstrated: Given any nondecreasing nonconstant function $\phi(t)$, $0\leqq t\leqq 1$, with $\phi(0)=0$, f is uniquely represented in the form $f(t)=k(t)+d(t)$, $0\leqq t\leqq 1$, so that $d(0)=0$, and ϕ is COV to k, and ϕ and $V_{0,t}d$ are NCAV. If f is nondecreasing, so are k and d. Given a function $f(t)$, $0\leqq t\leqq 1$, of bounded variation, substitutions $x=\phi(t)$ are determined which yield functions $f\phi^{-1}(x)$ which are absolutely continuous. Various applications are discussed. For completely additive set functions in an abstract space, the concepts of COV and CAV are shown to correspond to absolute continuity and singularity respectively.

P. V. Reichelderfer (Columbus, Ohio).

Viola, Tullio. Sul comportamento d'una porzione di superficie regolare, in prossimità del contorno. Rend. Sem. Mat. Univ. Padova **20**, 1–23 (1951).

Example of a surface S bounded by a rectifiable contour C where C contains a point P such that it is impossible to join, by a curve of finite length on $S-C$, two near points $P'P''$ of C on either side of P. *L. C. Young.*

Santaló, L. A. Some integral formulas and a definition of q-dimensional area for a set of points. Univ. Nac. Tucumán. Revista A. **7**, 271–282 (1950). (Spanish. English summary)

The author's English summary treats only his proposed definition of q-dimensional area of a set of points. However, the paper itself is mainly concerned with formulae, obtained by simple formal calculations which make implicit use of the usual assumptions of elementary differential geometry. The author's main formula connects the elementary q-dimensional area $|S|$ (of a sufficiently smooth q-dimensional variety S without multiple points, situated in n-space E_n) with the elementary $(q-h)$-dimensional area $|S(P)|$ of the intersection $S(P)$ of S by h varieties of dimension $n-1$ which depend on a parameter P. In the most interesting case, these varieties of dimension $n-1$ are unit spheres and P is the ordered set of their centres $P_1,\cdots,P_h$: The formula is then (I) $\int|S(P)|\,dP=\beta|S|$ with

$$\beta=\beta(n,q,h)=2^h\pi^{\frac{1}{2}nh}[g(n)]^{-h}g(q)[g(q-h)]^{-1},$$

where $g(z)=\Gamma[\frac{1}{2}(z+1)]$. It is assumed that $h\leqq q$ and in the case $h=q$ that $|S(P)|$ means the number of points of $S(P)$.

The author observes that when S is an arbitrary analytic set in E_n the formula (I) with $h=q$ provides a certain definition of q-dimensional area of the set S, and that this definition then automatically agrees with the elementary defini-

tion in sufficiently simple cases. However he does not attempt to relate it in any more general way to other non-elementary definitions of area in use at the present time.

L. C. Young (Madison, Wis.).

Valle Flores, E. On the extension of the theory of Lebesgue area to surfaces imbedded in R_n. Bol. Soc. Mat. Mexicana 6, 1–26 (1949). (Spanish)

The author extends to surfaces imbedded in a Euclidean space R_n various statements regarding the Lebesgue area proved by T. Radó [Trans. Amer. Math. Soc. 51, 336–361 (1942); these Rev. 3, 229] and J. W. T. Youngs [Bull. Amer. Math. Soc. 51, 669–673 (1945); these Rev. 7, 283] for $n=3$. The extensions are obtained by only formal modifications of the proofs given in the above mentioned papers.

L. Cesari (Lafayette, Ind.).

Pi Calleja, Pedro. Length and area. Univ. Nac. Tucumán. Revista A. 7, 157–242 (1950). (Spanish)

Expository article "summarizing" (in considerable detail) results and methods taken from Radó's book [Length and area, Amer. Math. Soc. Colloquium Publ., v. 30, New York, 1948; these Rev. 9, 505], from a few lectures by L. Cesari, and from recent papers. The article does not quite cover the state of our knowledge of the subject at the present time; in particular, there is no mention of the work of Choquet and his collaborators, work which unfortunately is still unpublished.

L. C. Young (Madison, Wis.).

Poplavskaya, G. Ya. The equivalence of various definitions of area of a continuous surface $z=f(x, y)$. Doklady Akad. Nauk SSSR (N.S.) 77, 21–24 (1951). (Russian)

For Borel subsets M of a nonparametric surface of finite Lebesgue area $z=f(x, y)$ situated in three-dimensional space, Verčenko [Rec. Math. [Mat. Sbornik] N.S. 10(52) 11–32 (1942); these Rev. 4, 154] introduced a surface measure which agrees with the Lebesgue area when M is the whole given surface. In the present paper, it is proved that Verčenko's surface measure agrees with the two-dimensional measures introduced by Carathéodory and by Hausdorff.

L. C. Young (Madison, Wis.).

Ewing, George M. Surface integrals of the Weierstrass type. Duke Math. J. 18, 275–286 (1951).

The Weierstrass (W-) integral for parametric curves and surfaces has the great advantage over the customary one defined in terms of Lebesgue integral, that it can be formulated for any representation of the curve or surface. A concept of W-integral for all surfaces S in E_3 of finite Lebesgue area $L(S)$ and for every representation was studied by the reviewer [Ann. Scuola Norm. Super. Pisa (2) 13, 77–117 (1948); 14, 47–79 (1948); these Rev. 9, 505; 10, 259]. For a particular type of representations (admissible representations) the author defines a W-integral which is proved to coincide with the one above. An interesting feature of the paper is the axiomatic character with which the main quantities are introduced (instead of a direct definition as is usually done), i.e., a scalar $A(S)$ (area) and a vector $V(S)$ (vector area) reflecting the global orientation of the surface. The axioms chosen for $A(S)$ are well known and characterize the Lebesgue area [T. Radó, Length and Area, Amer. Math. Soc. Colloq. Publ., v. 30, New York, 1948, p. 554; these Rev. 9, 505]; hence $A(S)=L(S)$. The axioms chosen for $V(S)$ state that (i) for every oriented triangle D, $V(D)$ has the magnitude $A(D)$ and the direction of the positive normal to D; (ii) for every oriented polyhedral surface P, $V(P)$ is the sum $\sum V(D)$ for all nondegenerate faces D of P; (iii) $P_n \to S$, $A(P_n) \to A(S)$, implies $V(P_n) \to V(S)$. These axioms define $V(S)$ uniquely. Their coherence and the existence of $V(S)$ are assured by expressing $V(S)$ in terms of the W-integral studied by the reviewer. By admissible representations of a surface S are meant the vector functions $x=x(u)$, $u \in R$ (R Jordan region) such that (i) given $\epsilon>0$ there are finite subdivisions τ of R into Jordan regions R_i with $n(\tau)<\epsilon$, $n(\tau)=$ norm of $\tau=\max_i \operatorname{diam}(R_i)$; (ii) $A(S)=\sum A(S_i)$, S, S_i being the surfaces defined by $x(u)$ on R, R_i. By a result of the reviewer all surfaces of finite Lebesgue area possess admissible representations. Let $f(x, u)$ be a function of the two vectors x, u which is continuous for all x and u and $f(x, ku)=kf(x, u)$, $k>0$. Let $x(u)$, $u \in R$, be an admissible representation of a surface S with $L(S)<+\infty$ and let τ be one of the above considered subdivisions of R. The author considers the sum $s=\sum f[x(u_i), V(S_i)]$ for an arbitrary choice of the points $u_i \in R_i$. The W-integral $J(S)$ is the limit of s as $n(\tau) \to 0$. $J(S)$ is proved to be independent of the admissible representation $x(u)$ of S and, for more restricted representations, it is proved that $S_n \to S$, $A(S_n) \to A(S)$ implies $J(S_n) \to J(S)$.

L. Cesari (Lafayette, Ind.).

Heisel, R. G. Remarks on the isoperimetric inequality. Duke Math. J. 18, 385–390 (1951).

The spatial isoperimetric inequality established by Radó [Trans. Amer. Math. Soc. 61, 530–555 (1947); these Rev. 9, 137] connects the Lebesgue area of an oriented closed surface S with the enclosed volume $V(S)=\int\int\int i\,dxdydz$ where $i=i(x, y, z, S)$ is the topological index of the point x, y, z with respect to S. Radó's proof uses only a semi-continuity property of $V(S)$ which remains valid for $V^*(S)=\inf\{\liminf V(P_n)\}$, where the infimum is taken with respect to all sequences of polyhedra P_n which converge in area to S. The present author remarks that $V^*(S) \geqq V(S)$ with equality if the set of points occupied by S is of three-dimensional measure zero, and he constructs an example of a surface for which $V^*(S)>V(S)$.

L. C. Young (Madison, Wis.).

Cecconi, Jaurès. Proprietà integrali delle trasformazioni piane. Rivista Mat. Univ. Parma 1, 439–447 (1950).

Let T be a continuous transformation from the unit square Q in the w-plane into the z-plane. For each simple polygonal region p in Q, let $G_k(p)$ denote the measure of the set of points z where the essential multiplicity function $\kappa(z, T, p)$ is not zero [see Radó, Duke Math. J. 14, 587–608 (1947); these Rev. 9, 231], and let $\underline{D}_k(w)$ be the lower derivative of $G_k(p)$ at each interior point w of Q. Denote by $E(T, p)$ the set of those points in Q each of which belongs to an essential maximal model continuum lying in the interior of p, and set $m(p)$ equal to the least upper bound of the measure $|T[P \cdot E(T, p)]|$ of the image under T of the set $P \cdot E(T, p)$ for all perfect sets P in p having measure zero. The author shows that $G_k(p) \leqq \int\int_p \underline{D}_k(w)+m(p)$, which he uses to derive the following conclusions. A necessary and sufficient condition in order that T be BV (of bounded variation in the Cesari sense, or equivalently, of essential bounded variation) is the existence of a finite positive constant M such that for every finite sequence of pairwise disjoint polygonal regions p_i in Q one has $\sum G_k(p_i)<M$. A necessary and sufficient condition in order that T be BV is that $\underline{D}_k(w)$ be summable in Q and that there exist a finite positive constant M such that for every perfect set P in Q having measure zero and every finite sequence of pairwise disjoint polygonal

regions p_i in Q one has $\sum |T[P\cdot E(T, p_i)]| < M$. A necessary and sufficient condition in order that T be AC (absolutely continuous in the Cesari sense, or equivalently, essentially absolutely continuous) is that $\underline{D}_k(w)$ be summable in Q and that for every perfect set P in Q having measure zero, $|T[P\cdot E(T, Q)]| = 0$. In particular, if T is BV, then a necessary and sufficient condition in order that T be AC is that for every perfect set P in Q, $|T[P\cdot E(T, Q)]| = 0$.

P. V. Reichelderfer (Columbus, Ohio).

Cecconi, Jaurès. Una proprietà caratteristica delle trasformazioni assolutamente continue. Rivista Mat. Univ. Parma 1, 433–438 (1950).

Let T be a continuous transformation from the unit square Q in an Euclidean plane into an Euclidean plane which is of bounded variation in the Cesari sense, or equivalently, of essential bounded variation [see Radó, Duke Math. J. 14, 587–608 (1947); these Rev. 9, 231]. Denote by E the set of those points in Q each of which belongs to an essential maximal model continuum in Q. The author proves that a necessary and sufficient condition in order that T be absolutely continuous in the Cesari sense (or equivalently, essentially absolutely continuous) is that for every set I in Q for which the measure of $E\cdot I$ is zero it is true that the set $T(E\cdot I)$ also has measure zero. *P. V. Reichelderfer.*

Tsuji, Masatsugu. Change of variables in the multiple Lebesgue integral. J. Math. Soc. Japan 2, 48–56 (1950).

This note gives proofs of: (a) the theorem that a topological transformation $u = f(x)$ of a domain D of n-space which satisfies the condition $\limsup_{h\to 0} \|f(x+h) - f(x)\|/\|h\| < \infty$ everywhere on D is a measurable transformation; and (b) the theorem of Rademacher [Math. Ann. 79, 340–359 (1919)] on the transformation of multiple integrals.

L. M. Graves (Chicago, Ill.).

Theory of Functions of Complex Variables

Szele, Tibor. On a vector-algebraic introduction of the complex numbers. Mat. Lapok 1, 349–362 (1950). (Hungarian. Russian and English summaries)

Expository lecture. *P. R. Halmos* (Montevideo).

Zeuli, Tino. Sulla continuazione analitica delle funzioni associate ai sistemi fisici lineari. Boll. Un. Mat. Ital. (3) 5, 247–251 (1950).

Another procedure, similar to that obtained by G. Zin [Nuovo Cimento (9) 6, 531–542 (1949); these Rev. 11, 649], is obtained for continuing an analytic function from a segment of the imaginary axis into the right half plane.

R. C. Buck (Madison, Wis.).

Val'fiš, A. Z. On a theorem of Pólya converse to a theorem of Fabry. Soobščeniya Akad. Nauk Gruzin. SSR 8, 197–204 (1947). (Russian)

The theorem in question states that if n_k are integers which increase but have $\liminf n_k/k < \infty$, then there is a power series $\sum a_k z^{n_k}$ which has a regular point on its circle of convergence. The author gives a simplified version of the proof given by Erdös [Trans. Amer. Math. Soc. 57, 102–104 (1945); these Rev. 6, 148], with numerical values for all the constants involved and detailed presentation of all computations. *R. P. Boas, Jr.* (Evanston, Ill.).

Martin, Yves. Sur un théorème de M. Radstrom. C. R. Acad. Sci. Paris 232, 136–137 (1951).

Let $S(g)$ be the set of points which are limit points of zeros of successive derivatives of g. The author states the following. Let $f(z) = \sum b_n z^n/n!$ and suppose that $\limsup |b_{n+1}/b_n| = D > 0$. Then, given $a \neq 1$ one can choose numbers w_n taking only the values 1 or a, such that $k(z) = \sum w_n b_n z^n/n!$ has a point of $S(k)$ inside the circle $|z| < K(a)/D$. An explicit value for $K(a)$ is given. If $D = 0$, then for no choice of w_n obeying $\lim (w_{n+1} b_{n+1}/w_n b_n) = 0$ does $D(k)$ contain a finite point. The theorem of Radstrom is concerned with $|w_n| = 1$. *R. C. Buck* (Madison, Wis.).

Ilieff, Ljubomir. Über die singulären Stellen an der Peripherie des Konvergenzkreises einer Potenzreihe. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. 41, 31–42 (1945). (Bulgarian. German summary)

By a sequence (γ) is meant any sequence of complex numbers $\gamma_0, \gamma_1, \gamma_2, \cdots$, each of which is equal to one of a finite number $k \geqq 2$ of distinct complex numbers $\delta_1, \delta_2, \cdots, \delta_k$. The author defines a power series to be of class $S(\alpha)$, for any real α, if it can be represented in the form $\sum_{\nu=0}^{\infty} \gamma_\nu c_\nu z^\nu$, where the γ_ν are elements of a sequence (γ) and the c_ν satisfy the two conditions:

$$\lim_{n\to\infty} c_{n+1}/c_n = 1, \quad 0 < \liminf_{n\to\infty} |c_n|/n^\alpha \leqq \limsup_{n\to\infty} |c_n|/n^\alpha < \infty.$$

Following closely an argument given by Szegö [S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. 1922, 88–91], the author obtains the following extension of Szegö's theorem: A power series of class $S(\alpha)$ can be continued analytically beyond the unit circle $|z| < 1$ only if the sequence (γ) is periodic from a certain index on. Let s be some fixed integer with $1 \leqq s \leqq k$ and λ any positive integer. Denote by $n_s(\lambda)$ the number of those γ in the set $\gamma_0, \gamma_1, \cdots, \gamma_\lambda$ which are unequal to δ_s. If $\lim_{\lambda\to\infty} \lambda/n_s(\lambda) = \infty$, it is shown that the corresponding series of class $S(\alpha)$ can not be continued beyond $|z| < 1$. For series of class $S(\alpha)$ this contains Fabry's gap condition if $\delta_s = 0$ for some s. *W. Seidel* (Rochester, N. Y.).

Dörge, Karl. Entscheidung des algebraischen Charakters von Potenzreihen mit algebraischen Koeffizienten auf Grund ihres Wertevorrates. Math. Ann. 122, 259–275 (1950).

Let the Puiseux series

$$\varphi(z) = a_k z^{k/q} + a_{k-1} z^{(k-1)/q} + \cdots + a_0 + a_{-1} z^{-1/q} + \cdots$$

be given, where q is a positive integer, k any integer, and $z^{1/q}$ is that qth root of z the absolute value of whose amplitude is least. The author investigates conditions under which $\varphi(z)$ is an algebraic or transcendental function. The notion of the degree of transcendentality of a complex number z plays an important part here and is defined as follows. If z is an algebraic number and $P(x)$ that irreducible polynomial which vanishes at z, possesses relatively prime integral coefficients, and has a positive leading coefficient, the height $\omega(z)$ of z is defined to be the maximum of the moduli of the coefficients of $P(x)$. By the exponent $\eta(z)$ of z is meant the expression $(\log \omega(z))\,|\log|z||^{-1}$, provided $z \neq 0$ and $|z| \neq 1$. The excluded numbers z are called singular. If z is nonsingular, its degree of transcendentality $\tau(z)$ is defined by $\max(\gamma(z), \eta(z))$, where $\gamma(z)$ is the order of z; if z is singular, any real number $\geqq 1$ may be assigned to it as its degree of transcendentality; finally, if z is transcendental, $\tau(z) = \infty$. The conditions under which $\varphi(z)$ is an algebraic or transcendental function when the a_ν $(\nu = k, k-1, \cdots)$ are

algebraic numbers are too complicated to be stated here in detail. Under additional restrictions on the Puiseux series, however, the algebraic or transcendental character of $\varphi(z)$ may be inferred from the boundedness or unboundedness, respectively, of the set of degrees of transcendentality of the values assumed by $\varphi(z)$ on certain sequences of points which tend to infinity. *W. Seidel* (Rochester, N. Y.).

Makar, Ragy H. Induced infinite matrices and induced basic sets of polynomials. Proc. Math. Phys. Soc. Egypt 4, no. 1, 61–69 (1949).

Let $\{p_n(z)\}$ be a simple basic set of polynomials [for terminology cf. J. M. Whittaker, Sur les séries de base de polynomes quelconques, Gauthier-Villars, Paris, 1949; these Rev. 11, 344]. The author proves that if the given set is effective on $|z|=R>0$ then the sets whose coefficient matrices are the induced matrices of the coefficient matrix of the given set are effective on $|z|=\max(R, 1)$ but not necessarily in a smaller circle. The induced matrix is defined as for finite matrices [J. H. M. Wedderburn, Lectures on Matrices, Amer. Math. Soc. Colloquium Publ., v. 17, New York, 1934, p. 75] with an additional rule to make the positions of its elements definite. *R. P. Boas, Jr.*

Spitzbart, A. On the minimum of a certain integral. Proc. Amer. Math. Soc. 2, 246–253 (1951).

The minimum $I(A, \alpha)$ of $\int_0^{2\pi}|f(e^{i\theta})|\,d\theta$ is obtained for a function $f(z)$ regular in $|z|\leqq 1$ and such that $f(\alpha)=A$, $f'(\alpha)=1$ ($|\alpha|<1$) [S. Kakeya, Proc. Phys. Math. Soc. Japan (3) 3, 48–58 (1921) for the case $\alpha=0$]. The minimum of $I(A, \alpha)$ as A varies is also calculated and is the minimum of the integral subject to $f'(\alpha)=1$, thus giving a more explicit form of an inequality discussed by the reviewer and Rogosinski [Acta Math. 82, 275–325 (1950), eqs. (8, 3, 8) and (9, 1, 5); these Rev. 12, 89]. *A. J. Macintyre.*

Hayman, Walter K. Sur le module des fonctions entières. C. R. Acad. Sci. Paris 232, 591–593 (1951).

For an entire function $f(z)$, set

$$\sigma(z)=\log|f(z)|+K\log M(r).$$

Results of Wiman [Acta Math. 41, 1–28 (1918)] and Beurling [Duke Math. J. 16, 355–359 (1949); these Rev. 10, 692] discuss the behavior of $\sigma(z)$ for large z and constant K, one for $f(z)$ without zeros, and the other for z on a fixed ray. The author takes $K=C\log\log\log M(r)$ and proves that there is a constant A_0 such that if $C>A_0$, then $\sigma(z)>0$ holds everywhere on $|z|=r$ for arbitrarily large r, but that this fails if $C<A_0$. *R. C. Buck* (Madison, Wis.).

Bowen, N. A., and Macintyre, A. J. Some theorems on integral functions with negative zeros. Trans. Amer. Math. Soc. 70, 114–126 (1951).

Let $0<a_1<a_2<\cdots$, and set $f(z)=\pi^{-1}\sin\pi\rho\prod(1+z/a_n)$ where $\rho<1$. The authors study the asymptotic properties of $f(z)$. Set $H(\theta)=\limsup_{r\to\infty} r^{-\rho}\log|f(re^{i\theta})|$ and

$$h(\theta)=\liminf_{r\to\infty} r^{-\rho}\log|f(re^{i\theta})|.$$

Then, it is proved that if $|\alpha|>\pi/2\rho$ then $H(\alpha)\leqq\cos\rho\alpha$ implies $h(0)\geqq 1$ and $h(\alpha)\geqq\cos\rho\alpha$ implies $H(0)\leqq 1$. If $H(\pi)\leqq\cos\pi\rho$ and if $\log|f(-x_n)|\geqq x_n^{\rho}(\cos\pi\rho-\epsilon)$ for a sequence $x_n\to\infty$ with $x_{n+1}/x_n\to 1$ then $f(x)\sim x^{\rho}$ as $x\to+\infty$, provided only that $\frac{1}{2}<\rho<1$. Titchmarsh [Proc. London Math. Soc. (2) 26, 185–200 (1926)] proved this for $0<\rho<\frac{1}{2}$ with the additional hypothesis $f(x)=O(\sqrt{x})$. The authors also prove similar theorems for functions f of higher order. *R. C. Buck* (Madison, Wis.).

Bowen, N. A., and Macintyre, A. J. An oscillation theorem of Tauberian type. Quart. J. Math., Oxford Ser. (2) 1, 243–247 (1950).

Let $f(z)=\prod_{n=1}^{\infty}(1+z/a_n)$ be an entire function of order ρ, with $0<\rho<1$, whose zeros are all negative. If

$$0<\lambda\leqq\liminf x^{-\rho}\log f(x)<\limsup x^{-\rho}\log f(x)\leqq\Lambda<\infty,$$

then there exist positive functions $\varphi(\lambda, \Lambda)$ and $\Phi(\lambda, \Lambda)$ such that, as $n\to\infty$,

$$\varphi(\lambda, \Lambda)\leqq\liminf n^{-1/\rho}a_n\leqq\limsup n^{-1/\rho}a_n\leqq\Phi(\lambda, \Lambda)$$

and $\Phi(\lambda, \Lambda)/\varphi(\lambda, \Lambda)$ is arbitrarily near unity provided that Λ/λ is sufficiently near unity. Best functions φ and Φ are conjectured with supporting arguments. *R. P. Angew.*

Vermes, P. Note on a two-point boundary problem. II. Quart. J. Math., Oxford Ser. (2) 2, 74–80 (1951).

The author continues part I [same J. (1) 19, 109–116 (1948); these Rev. 9, 577]. With the notation used there, the author now gives order conditions on a_i and b_j which ensure the existence of an entire function with given sequences of derivatives taking these values at 0 and 1. With a further restriction on the sequences of assigned derivatives, an order condition ensures the existence of a unique solution of exponential type. Further extensions deal with the many-point instead of the two-point case. *R. P. Boas, Jr.*

Pham, Tinh-Quat. Quelques propriétés des fonctions méromorphes périodiques. Ann. Sci. École Norm. Sup. (3) 67, 307–320 (1950).

Making use of results in a previous paper [same Ann. (3) 65, 11–70 (1948); these Rev. 10, 363] the author considers meromorphic functions $f(z)/g(z)$ where f and g are finite order entire functions each periodic with period 2π. In this class, he studies the form taken by the characteristic function of Nevanlinna, and corresponding theorems concerning defect and Picard exceptional values. *R. C. Buck* (Madison, Wis.).

Valiron, G. Sur l'interpolation par les fonctions méromorphes. Proc. Math. Phys. Soc. Egypt 4, no. 1, 23–26 (1949).

Let $\{a_n\}$ be a sequence of finite order ρ. Under certain conditions an entire function $f(z)$ of order ρ exists taking on given values A_n at a_n. The author shows that further conditions on a_n and A_n are unnecessary in the solution of the problem of interpolation $f(a_n)=A_n$ when the entire function $f(z)$ is replaced by a suitable meromorphic function of order ρ. The method of construction of this meromorphic function of order ρ is given. When the a_n and A_n are real, a real function $f(x)$ is constructed, analytic for every finite real value x, for which $f(a_n)=A_n$, and at the same time

$$\frac{1}{x}\int_{-x}^{x}|\log|f(x)||\,dx<x^{\rho+\epsilon},\qquad x>x_\epsilon>0,\qquad \epsilon>0.$$

M. S. Robertson (New Brunswick, N. J.).

Valiron, Georges. Sur les valeurs déficientes des fonctions algébroïdes méromorphes d'ordre nul. J. Analyse Math. 1, 28–42 (1951). (French. Hebrew summary)

Suppose that $u(z)$ is a p-valued meromorphic algebroid function of order zero and that $T(r, u)$, $N(r, a)$ are defined

in the usual way. Let $N(r, a_1, a_2, \cdots, a_{\nu+1}] = \max N[r, a_j]$ for $j=1$ to $\nu+1$. The author shows that for any $\nu+1$ distinct values a_j, (*) $N[r, a_1, a_2, \cdots, a_{\nu+1}]/\nu T[r, u] \to 1$ as $r \to \infty$ through a sequence of values $r=r_n$. Hence $u(z)$ has at most ν deficient values in the sense of Nevanlinna. If further $T(r)$ is of sufficiently regular growth or $T(r)=O(\log r)^2$, (*) holds generally as $r \to \infty$. If in this case $u(z)$ has ν deficient values then any other value a is normal, i.e., $N(r, a) \sim \nu^{-1} T(r)$, as $r \to \infty$. In the above results the values a_j can also be replaced by functions $f_j(z)$ of slower growth than $u(z)$. The proofs follow from asymptotic equalities of the type $\log|u(z)| \sim N[r, 0]$ subject to certain conditions as $|z|=r \to \infty$ established in the case $\nu=1$ by the author [Ann. Fac. Sci. Univ. Toulouse (3) 5, 117–257 (1913); Bull. Sci. Math. (2) 71, 25–32 (1947); these Rev. 9, 342].

W. K. Hayman (Exeter).

Radojčić, M. Une proposition sur les singularités essentielles des fonctions analytiques. Acad. Serbe Sci. Publ. Inst. Math. 3, 137–142 (1950).

The paper contains a contribution to the theory of "fundamental regions" of an analytic function in the neighborhood of an isolated singular line. The result applies to a restricted class of functions and asserts that the diameters of the fundamental regions tend to zero. *L. Ahlfors.*

Moneo, Alberto. On a class of birational transformations in the complex plane. Gaceta Mat. (1) 2, 191–196 (1950). (Spanish)

The author discusses the geometric properties of the "semihomographic" transformation $w=(az+b)/(cz+d)$.

R. C. Buck (Madison, Wis.).

Olguín, Juan. On approximation of the second order in the neighborhoods of a pair of corresponding points in a conformal transformation. Math. Notae 9, 118–129 (1949). (Spanish)

The approximation is made by a suitable homographic transformation. In place of curves through the corresponding points, their osculating circles are studied by geometrical arguments based on inversion. In particular the theorem of L. A. Santalo [Math. Notae 5, 29–40 (1945); these Rev. 6, 261] that the correspondence of centres of curvature is a projectivity is proved. *A. J. Macintyre.*

Opitz, G. Zur Konvergenz bei genäherter konformer Abbildung. Z. Angew. Math. Mech. 30, 337–346 (1950). (German. English, French, and Russian summaries)

A method for replacing the Theodorsen-Garrick integral equation for the mapping function on the boundary by a system of simultaneous nonlinear equations to be used in approximating the solution of the Theodorsen-Garrick equation by iteration is given in this paper. The convergence "in the mean" of the iteration procedure for the system of simultaneous equations is proved and it is shown that as the number of equations increases the solution of the sytem of simultaneous equations converges to the solution of the integral equation. The formulation of the above system of simultaneous equations was published in a paper by I. Naiman [Nat. Adv. Comm. Aeronaut. Wartime Rep. L-153 (1945)]. The above convergence theorems as well as certain uniform pointwise convergence theorems were given by the reviewer [Bull. Amer. Math. Soc. 56, 177 (1950)].

C. Saltzer (Cleveland, Ohio).

Seth, B. R. Boundary conditions interpreted as conformal transformation. Proc. Amer. Math. Soc. 2, 1–4 (1951).

The author makes the obvious remark that many boundary value problems to which function theory is applicable can be formulated as problems in conformal mapping, and demonstrates for certain general types of boundary conditions where polygonal boundaries are involved that the problems can be solved (with help of Schwarz-Christoffel) in essentially closed form. For curved boundaries the method is reduced to that for polygonal boundaries by approximation. *D. Gilbarg* (Bloomington, Ind.).

Komatu, Yûsaku. Representation of functions analytic in a multiply-connected domain. Kôdai Math. Sem. Rep. 1950, 69–72 (1950).

Let D denote the canonical domain of multiplicity n (>2) consisting of an annular ring $Q<|z|<1$ slit along concentric circular arcs $|z|=m_j, 3 \leqq j \leqq n, \theta_j \leqq \arg z \leqq \theta_j+\gamma_j$. Let $C=\sum C_j$ be the boundary of D, consisting of the two circles and $n-2$ circular arcs. Let $f(z)$ be single-valued and regular in D and piecewise regular on $D+C$, having a finite number of boundary or exceptional points $z_{j\mu}$, ($\mu=1, 2, \cdots, n_j$; $j=1, \cdots, n$) where $f(z)$ is not regular. It is assumed that $f'(z)$ has limits from both sides along C at the exceptional points and that $f'(z)$ does not vanish on $D+C$ except possibly at $z_{j\mu}$. A representation for $f(z)$ in D is obtained in the form

$$1+\frac{zf''(z)}{f'(z)}=\frac{1}{2\pi}\sum_{j=1}^{n} m_j \int_{C_j}\left(\frac{\partial G(\zeta, z)}{\partial \nu_\zeta}+\Re M_j(\zeta)\right) d \arg df(\zeta),$$

where $G(\zeta, z)$ is an analytic function of z whose real part is the Green function of D, and where on C_j

$$M_j(\zeta)=M(\log z+2\pi i, \zeta)-M(\log z, \zeta),$$

$$M(\log z, \zeta)=L(z, \zeta)=\frac{1}{2\pi}\int_{z_0}^{z}\frac{\partial}{\partial \nu_\zeta}G(\zeta, z)\frac{dz}{z},$$

ν_ζ being the inward normal parameter at a boundary point ζ. This formula is an extension from the cases $n=1$ and 2 obtained earlier by the author [see Proc. Imp. Acad. Tokyo 20, 536–541 (1944); Jap. J. Math. 19, 203–215 (1945); these Rev. 7, 287]. An application of the general formula is made to the case where $w=f(z)$ maps the basic domain D onto a domain bounded by n rectilinear polygons.

M. S. Robertson (New Brunswick, N. J.).

Beurling, Arne. An extremal property of the Riemann zeta-function. Ark. Mat. 1, 295–300 (1951).

For $k>0$ consider the class C_k of all Dirichlet's series $\varphi(s)=\sum \lambda_n^{-s}$ with $0<\lambda_n \leqq \lambda_{n+1}$ such that the series converges for $\sigma>1$ and $\varphi(s)$ has the properties (α) $\varphi(s)-(s-1)^{-1}$ is entire, (β) $\varphi(-2n)=0$, $n=1, 2, 3, \cdots$,

$$(\gamma) \qquad \max|\varphi(re^{i\theta})|<C\Gamma(r)(2\pi k)^{-r}$$

for $r>2$. Let $D_k=\bigcap_{\epsilon>0} C_{k-\epsilon}$. Then for $0<k \leqq \frac{1}{2}$ each class D_k contains infinitely many elements while all classes D_k with $k>\frac{1}{2}$ are vacuous except for D_1 which consists of two elements $\zeta(s)$ and $(2^s-1)\zeta(s)$. The proof is based on the Phragmén-Lindelöf principle which yields two lemmas. (I) $\varphi(s) \in D_k$ if and only if the entire function $f(z)=\prod_1^\infty(1+z^2\lambda_n^{-2})$ satisfies $f(x)=a|x|^p e^{\pi|x|}+O[e^{\pi(1+\epsilon-2k)|x|}]$ for fixed a and p, $a>0$, p real, and all large real x. (II) If $f(z)$ is entire, $f(0)=1$, and $f(x)=a|x|^p e^{\pi|x|}+O[e^{-\delta|x|}]$ and if $f(z)$ is of order one and normal type, then $f(z)$ is either $\cosh \pi z$ or $\sinh \pi z/(\pi z)$.

E. Hille (New Haven, Conn.).

Leja, F. Sur les coefficients des fonctions analytiques univalentes dans le cercle et les points extremaux des ensembles. Ann. Soc. Polon. Math. 23, 69–78 (1950).

Let $y=f(x)=\sum_1^\infty c_n x^n$, with c_n real and positive, be a function holomorphic and univalent in the circle K ($|x|<1$). Denote by Δ the domain covered by y when x varies in K, by Γ the boundary of Δ, and let D and F be the images of Δ and Γ under the transformation $z=1/y$. The author has succeeded in expressing the coefficients c_n exclusively in terms of the transfinite diameter $d(F)$ of the bounded continuum F and of the limits of the arithmetic means of the extremal points of F. The connection with the well-known results of Bieberbach and Löwner is pointed out.

More precisely, let $\eta_{1n}, \eta_{2n}, \cdots, \eta_{nn}$ be the values taken by n points $z_1, z_2, \cdots, z_n$ of F chosen so as to make $\prod_{1\leq j<k\leq n}(z_j-z_k)$ a maximum. Write $\Delta_j(\eta_{1k}, \cdots, \eta_{nk})$ for $\prod_{k=1}^n(\eta_{jn}-\eta_{kn})$, $j\neq k$, and suppose that the η_{in} are so ordered that the sequence $\Delta_j(\eta_{1n}, \cdots, \eta_{nn})$, $j=1, \cdots, n$, is non-decreasing. Now write Δ_{n-1} for $\Delta_1(\eta_{1n}, \cdots, \eta_{nn})$. The author has proved [same Ann. 12, 57–71 (1934); 18, 4–11 (1945); these Rev. 8, 255] that $d(F)=\lim_{n\to\infty}(\Delta_{n-1})^{1/(n-1)}$. Here he also proves that $\lim_{n\to\infty}(\eta_{1n}^p+\cdots+\eta_{nn}^p)/n=s_p$, $p=1, 2, \cdots$, all exist and effects the calculations $c_1d=1$, $c_2d^2=-s_1$, $2c_3d^3=(3s_1^3-s_3)$, $3c_4d^4=-(8s_1^3-6s_1s_2+s_3)$, $(d=d(F))$, using the auxiliary function $g(y)=\lim_{n\to\infty}|\prod_{j=1}^n(z-\eta_{jn})|^{-1/n}$ for which it is proved that $x=g(y)d(F)=f^{-1}(y)$. There are a number of minor misprints including the occasional omission of moduli signs. *R. Wilson* (Swansea).

Goodman, A. W., and Robertson, M. S. A class of multivalent functions. Trans. Amer. Math. Soc. 70, 127–136 (1951).

For $p\geqq 1$ let $T(p)$ be the class of all functions $f(z)=\sum_1^\infty b_k z^k$ which are regular in $|z|<1$, have real coefficients b_k, and for which $\Im f(z)$ changes sign exactly $2p$ times on every circle $|z|=r$ when r (<1) is sufficiently large. The class $T(1)$ is the class of typically-real $f(z)$. It is proved, by induction from the known results on $T(1)$, that for $n>p$

$$|b_n|\leqq\sum_{k=1}^{p}\frac{2k(n+p)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)}|b_k|,$$

and that this estimate is best possible. In fact, this is true for the subclass of p-valent functions with real coefficients which are "starlike with respect to a real point" [cf. Robertson, Duke Math. J. 12, 669–684 (1945); these Rev. 7, 379]. For p-valently convex functions $f(z)$ with real coefficients a (sharp) improvement of the above estimate for $|b_n|$ is also obtained. *W. W. Rogosinski* (Newcastle-upon-Tyne).

Lewy, Hans. Developments at the confluence of analytic boundary conditions. Univ. California Publ. Math. (N.S.) 1, 247–280 (1950).

Let $u+iv=w=F(z)$ be schlicht in the half-plane $y>0$ ($z=x+iy$) and map the half-plane onto a domain bounded by a simple closed curve Γ. Let $F(0)=0$ and let Γ be representable near $w=0$ as follows: $v=u^2f(u)$ for $u\leqq 0$, $v=u^2g(u)$ for $u\geqq 0$, where f and g are convergent real power series. It is then shown that $F(z)$ can be represented near $z=0$ by an asymptotic double series $\sum A_{mn}z^m(z\log z)^n$. A similar representation is shown to hold with a convergent double series if $F(z)$ is analytic for $y>0$, $|z|<k$ and $u=\Re(F(z))$ is continuous in this domain plus the segment $-k<z<k$, with $\partial u/\partial y=0$ for $z<0$ and $\partial u/\partial y=a(x)u$ for $z>0$, where $a(x)$ is a convergent real power series with nonvanishing constant term. In the second case the function F is uniform on the Riemann surface of $\log z$ for $|z|<k$ and satisfies the difference-differential equation: $F'(z+)-F'(z)+i(F(z+)+F(z))=0$, where z and $z+$ lie one above the other on successive sheets of the Riemann surface. The solutions of this equation are studied in detail.

W. Kaplan (Ann Arbor, Mich.).

Garabedian, P. R. Asymptotic identities among periods of integrals of the first kind. Amer. J. Math. 73, 107–121 (1951).

The paper deals with Poincaré's problem [J. Math. Pures Appl. (5) 1, 219–314 (1895)] of finding asymptotic identities among B-periods $\pi^2\rho_{\mu\nu}$ of the g linearly independent integrals of first kind $w_\nu(z)$ with A-periods $\delta_{\mu\nu}\pi i$ on a (degenerating) closed Riemann surface S of genus g. The author derives the $\frac{1}{2}(g-2)(g-3)$ possible identities under the following restrictive assumptions: (1) S is symmetric, created by doubling a plane domain D of finite connectivity bounded by n proper continua C_ν; (2) the B-periods are taken along C_μ; (3) the degeneration of D is as in the three cases below.

First, C_μ are circles centered at a_μ with radii $t\varepsilon_\mu$ ($\mu<n$) and $1/(t\varepsilon_n)$; the degeneration is created by letting $t\to 0$. Then, disregarding the terms of higher order,

$$p_{\mu\nu}=(\delta_{\mu\nu}-n^{-1})/|\log t|+\rho_{\mu\nu}/|\log t|^2$$

and the identities are

$$\sum\exp\left[2(2\rho_{ij}+\rho_{kl}+\rho_{lm}+\rho_{mk})\right]=\sum\exp\left[2(\rho_{ij}+\rho_{jk}+\rho_{kl}+\rho_{lm}+\rho_{mi})\right]$$

where i, j, k, l, m are distinct and range over any set of five integers between 1 and n. The relations are derived in a simple manner from identities among the distances between the a_μ, by means of the harmonic measure of C_μ in D. Secondly, $|z|=1$ is selected as the outer circle C_n. Then $p_{\mu\nu}=\delta_{\mu\nu}/|\log t|+P_{\mu\nu}/|\log t|^2$ and

$$\begin{vmatrix}(1-\delta_{\mu\nu})/(\varepsilon^{-P_{\mu\nu}}-1) & 1\\ 1 & -2\end{vmatrix}=0.$$

Thirdly, C_n is the real axis, C_μ ($\mu<n$) are horizontal line segments in $\Im(z)>0$ with a length $l_\mu t$ at a height $h_\mu t$ above C_n. For $\pi_{\mu\nu}=\lim_{t\to 0}p_{\mu\nu}t^{-2}$ the Poincaré identities are refound:

$$(\pi_{ij}\pi_{ik}\pi_{jl}\pi_{kl})^{\frac{1}{2}}+(\pi_{ik}\pi_{il}\pi_{jk}\pi_{jl})^{\frac{1}{2}}=(\pi_{ij}\pi_{il}\pi_{jk}\pi_{kl})^{\frac{1}{2}}.$$

The case where the A-periods are taken along C_μ is then discussed. The identities remain valid when the limiting behaviour of the periods is prescribed. The author remarks on the possibility of interpreting the invariants $\rho_{\mu\nu}$ as conformal moduli. *L. Sario* (Princeton, N. J.).

Fourès, Léonce. Sur la théorie des surfaces de Riemann. Ann. Sci. École Norm. Sup. (3) 68, 1–64 (1951).

This thesis is concerned with three subjects: 1) Bieberbach's theory of fusion (Verschmelzung, raccordement), 2) topological trees of certain covering surfaces, 3) division of a Riemann surface into sheets.

The method of fusion was introduced by Bieberbach [Lehrbuch der Funktionentheorie, v. 2, 2d ed., Teubner, Leipzig-Berlin, 1931] to perform the uniformization by geometric methods, without using harmonic functions. The author deals with the following general fusion problem: Let A, B be disjoint circular discs with subregions A', B', resp., in conformal correspondence $\zeta=\psi(z)$. The problem is to determine functions $w=f(z)$ and $w=\varphi(\zeta)$, meromorphic and univalent on A and B, coinciding at $z\in A'$, $\psi(z)\in B'$ and taking different values at any other couple $z\in A$, $\zeta\in B$. The

author performs the fusion in a number of different cases, depending on the conformal configurations of A' and B'. Boundary relations are thoroughly investigated. A second kind of fusion is treated. In the proofs, use is made of Montel's normal families and Bieberbach's distortion theorem. The results are applied to overlapping regions on abstract Riemann surfaces, defined by parameter discs. The classical uniformization theorems are reproduced. Covering surfaces, defined axiomatically, are studied in the special case with branch points over a finite number of projection points on the complex plane. In the class of regularly branched topological trees, a subclass is characterized where each surface, represented by the three, is a covering surface of another surface. Finally the domain of univalence of the inverse function of the uniformizer of a simply connected Riemann surface is studied. A method is indicated which consists of embedding the transcendental singularities in domains of decreasing diameters. *L. Sario.*

Tsuji, Masatsugu. A deformation theorem on conformal mapping. Tôhoku Math. J. (2) **3**, 7–12 (1951).

Let D be a simply connected domain of the z-plane containing $z=0$ and with $z=\infty$ on its boundary Γ consisting of a countable number of curves C, each extending to infinity in both directions. Let z_0, $|z_0|=r_0$, be the point on Γ lying closest to $z=0$, and let $\theta_r^{(i)}$ $(i=1, 2, \cdots, n(r))$ represent the cross cuts lying on $|z|=r$ and separating 0 from ∞ in D. Assume that $n(r)<\infty$, but possibly $n(r)\to\infty$ as $r\to\infty$. Let $\theta_r=\sum_{i=1}^{n(r)}\theta_r^{(i)}$, and $r\theta(r)=$ length of θ_r. The author proves the basic distortion theorem: If $w=f(z)$ maps D conformally onto $|w|<1$ with $f(0)=0$, then the image of θ_r in $|w|<1$ can be enclosed in a finite number of circles $K_r^{(i)}$, $i=1, 2, \cdots, \nu(r)\leqq n(r)$, which cut $|w|=1$ orthogonally such that the sum of their radii is less than

$$C\exp\left(-\int_{r_0}^{kr}\frac{dr}{r\theta(r)}\right),$$

where C and k are constants with $0<k<1$. The main interest in this theorem lies in the "countable number of contours," for if one has only one contour, the Ahlfors distortion theorem yields the sharper result with $k=1$ [Ahlfors, Acta Acad. Aboensis. 6, no. 9 (1932)]. Several theorems are then presented as applications of the basic distortion theorem.

G. Springer (Evanston, Ill.).

Tsuji, Masatsugu. Beurling's theorem on exceptional sets. Tôhoku Math. J. (2) **2**, 113–125 (1950).

An application of Beurling's general theorem on exceptional sets [Acta Math. 72, 1–13 (1939); these Rev. 1, 226] states: If $f(z)$ maps $|z|<1$ onto a Riemann domain of finite spherical area, the radial limit of $f(z)$ for $r\to 1$ exists except at most for a point set E on $|z|=1$ of vanishing outer capacity. The theorem was proved by showing that the function $F(r,\theta)=\int_0^r|\partial f(r,\theta)/\partial r|\,dr$ has the asserted radial property. The geometric interpretation of $F(1,\theta)$ leads the author to the following generalization: Under the assumption of Beurling's theorem there is a set E on $|z|=1$ of vanishing capacity such that a rectilinear segment joining a boundary point not in E with any point of $|z|<1$ is mapped onto a rectifiable curve on the w-sphere. If w is an ordinary value of a function $f(z)$ mapping $|z|<1$ onto a domain of finite spherical area, then [Beurling, loc. cit., p. 11] the set E_w on $|z|=1$ with radial limits w of $f(z)$ is of outer capacity zero. The author extends this theorem to the case where the function $f(z)$ assumes three values a finite number of times in $|z|<1$. If $f(z)$ maps $|z|<1$ onto a plane region D not containing the infinite point, and if E is a closed set of accessible boundary points of D with vanishing capacity, then the set on $|z|=1$, with radial limits in E, is of capacity zero. These statements are proved by using the well-known lemmas of Evans, Lindelöf, Montel and Fejer-Riesz.

L. Sario (Princeton, N. J.).

Tsuji, Masatsugu. A theorem on the majoration of harmonic measure and its applications. Tôhoku Math. J. (2) **3**, 13–23 (1951).

Consider in a subdomain D of $|z|<R$ the harmonic measure $u_R(z)$ of the intersection of $|z|=R$ and the boundary of D. Denote by $r\theta^i(r)$ the lengths of the arcs θ_r^i which constitute the part θ_r of $|z|=r$ contained in D. Assuming $z=0$ belongs to D, put $\tilde{\theta}(r)=\infty$ if $|z|=r$ lies entirely in D, otherwise $\tilde{\theta}(r)=\sup\theta^i(r)$; write $I(a,b)=\int_a^b(dr/r\tilde{\theta}(r))$. The author derives, for any $0<\alpha<1$, the following majoration of the harmonic measure: $u_R(0)\leqq(2e/(1-\alpha))^{\frac{1}{2}}\exp(-\pi I(0,\alpha R))$. An analogous estimation is given for the case in which the origin is not in D. Application is then made to a function $w(z)$, analytic and single-valued in an infinite domain D and on its boundary Γ such that $|w(z)|\leqq\lambda$ on Γ, $|w(z_0)|>\lambda$ at a point $z_0\in D(|z_0|=r_0)$. For the growth of $M(r)=\max|w(z)|$ on θ_r the inequality $\log\log(M(r)/\lambda)\geqq\pi I(r_0,\alpha r)-\text{const.}$ is given, in extension to Arima's theorem (to appear). Finally, a domain D in $|z|<R$ with $z=0$ on its boundary is studied. It is shown, in accordance with Beurling's theorem [Thesis, Uppsala, 1933], that $z=0$ is a regular point for the Dirichlet problem if $I(0,R)=\infty$. *L. Sario* (Princeton, N. J.).

Arima, Kihachiro. On harmonic measure functions in some regions. Kôdai Math. Sem. Rep. **1950**, 75–80 (1950).

In this note a particular type problem, for plane regions S bounded by $|z|=1$ and an infinite set of analytic curves in $|z|<1$, is dealt with. Denoting by $\omega(r)$ the harmonic measure of the intersection of $|z|=r$ and S, with respect to the intersection of $|z|<r$ and S, the problem is: When is a given S a "0-region," i.e. $\omega(r)\to 0$ for $r\to 1$? The author gives a number of sufficient conditions, in terms of the geometric configuration of S, and indicates applications to the fundamental regions of Fuchsoid groups. [This paper and the two papers reviewed below are not in final form, and will be published in more detail elsewhere]. *L. Sario.*

Arima, Kihachiro. On uniformizing functions. Kôdai Math. Sem. Rep. **1950**, 81–83 (1950).

A closed Riemann surface corresponding to the equation $S(x,y)=0$ is considered, $S(x,y)$ being a polynomial of degree m in y. If $f(t)$ and $g(t)$ are uniformizing meromorphic functions in $|t|<R$, $S(f(t),g(t))\equiv 0$, then the inequality $\limsup_{r\to R}T(r,f)/\log(1/(R-r))\leqq m/(2g-2)$ holds for the characteristic function $T(r,f)$ of $f(t)$. *L. Sario.*

Arima, Kihachiro. On a meromorphic function in the unit circle whose Nevalinna's characteristic function is bounded. Kôdai Math. Sem. Rep. **1950**, 94–95 (1950).

Sufficient conditions for the degeneration of a meromorphic function $f(z)$ with a bounded characteristic in $|z|<1$ are given in terms of the behaviour of $f(z)$ near a Borel set of positive capacity on $|z|=1$. *L. Sario.*

Hervé, Michel. Sur les fonctions fuchsiennes de deux variables complexes. C. R. Acad. Sci. Paris **232**, 673–675 (1951).

Let $z=(x, y)$ be a two-dimensional complex vector, D a domain in z-space, and $z'=f(z)=(f_1(x, y), f_2(x, y))$ an automorphism of D. Let γ denote a discontinuous group of automorphisms $f(z)$ with a fundamental domain, the closure of which belongs to D. A function $\theta_m(z)$ analytic in D, is called a Fuchsian function of dimension m, if

$$\theta_m(f(z))=\theta_m(z)[\partial(f_1, f_2)/\partial(x, y)]^{-m}$$

for every $f=(f_1, f_2)$ of γ. If $d(m)$ denotes the number of linearly independent functions $\theta_m(z)$, the ratio $d(m)/m^2$ will tend to a limit, when $m\to\infty$: The author outlines a proof of this theorem, based on the earlier result that $\limsup_{m\to\infty} d(m)/m^2$ is finite [same C. R. **226**, 462–464 (1948); these Rev. **9**, 343], and on recent investigations by H. Cartan [Bull. Soc. Math. France **78**, 29–64 (1950); these Rev. **12**, 172]. *H. Tornehave* (Copenhagen).

Theory of Series

Strang, J. A. On a class of recurrence relations. Communications Fac. Sci. Univ. Ankara **2**, 1–10 (1949).

In this paper the author is concerned with power series whose coefficients satisfy certain recurrence formulae. His purpose is to determine dominant functions for the power series. The paper is filled with the statement "we can prove" or its equivalent. The reviewer has no reason to doubt but that careful mathematical induction will establish these results. *T. Fort* (Athens, Ga.).

Milošević, Kovina. Sur un problème de Piza. Bull. Soc. Math. Phys. Macédoine **2**, 25–29 (1951). (Serbo-Croatian. French summary)

Let $S_1(n, k)=1^k+2^k+3^k+\cdots+n^k$, n and k being positive integers, and let

$$S_{p+1}(n, k)=S_p(1, k)+S_p(2, k)+\cdots+S_p(n, k).$$

The author shows that $S_p(n, k)$ is a polynomial in n of degree $p+k$. The formulas obtained are more involved and somewhat more explicit than those given contemporaneously for the problem of Pisa by Lessard [Amer. Math. Monthly **58**, 429–430 (1951)]. *R. P. Agnew.*

Bonferroni, Carlo. Una catena di criteri di convergenza per serie e integrali a termini positivi. Boll. Un. Mat. Ital. (3) **5**, 218–225 (1950).

The author shows how, starting with a particular case of Kummer's test for convergence or divergence of series, one can construct an infinite sequence of tests each more effective than the preceding. One such sequence has the ratio test, Raabe's test, and DeMorgan's test for its first three members. In a similar manner one can construct sequences of tests for convergence or divergence of improper integrals. *O. Frink* (State College, Pa.).

Schmetterer, L. Beitrag zur Multiplikation unendlicher Reihen. Monatsh. Math. **54**, 313–329 (1950).

Let $\sum c_n$ be the Cauchy product series of two series $\sum a_n$ and $\sum b_n$ which have real terms. Theorem 1 gives Tauberian conditions under which $r>0, s>0$, $C_r\sum a_n=A$ and $C_s\sum b_n=B$ imply $C_{r+s}\sum c_n=AB$. Theorem 2 gives conditions under which convergence of $\sum a_n$ and $\sum b_n$ implies that of $\sum c_n$. Theorems 3 and 4 give conditions under which $\lim_{n\to\infty}\sum_{k=-n}^{n}a_k=A$ and $\lim_{n\to\infty}\sum_{k=-n}^{n}b_k=B$ imply $\lim_{n\to\infty}\sum_{k=-n}^{n}d_k=AB$ where $\sum d_k$ is the Fourier product series for which $d_k=\frac{1}{2}\sum_{p+q=k}a_pb_q$. Theorem 5 gives conditions under which convergence of double series $\sum a_{mn}$ and $\sum b_{mn}$ to A and B respectively implies convergence of their Cauchy product series $\sum c_{mn}$ to AB. Theorem 6 shows that similar conditions are insufficient. Theorem 7 gives other similar conditions under which restricted convergence of $\sum a_{mn}$ and $\sum b_{mn}$ to A and B implies restricted convergence of the Cauchy product series $\sum c_{mn}$ to AB. *R. P. Agnew* (Ithaca, N. Y.).

Kritikos, N. Remarques sur la convergence ou la divergence de certaines suites. Bull. Soc. Math. Grèce **25**, 40–48 (1951). (Greek. French summary)

Let A_n be a sequence of points in Euclidean m space, satisfying the condition that for every $\epsilon>0$ there is a point A_{n_ϵ} such that the distances of all subsequent points from A_{n_ϵ} differ from each other by less than ϵ. Then the sequence either converges or contains a point from which all later points are equidistant. *R. P. Boas, Jr.*

Boas, R. P., Jr. A note on series of positive terms. J. Univ. Bombay (N.S.) **19**, part 3, sect. A, 12 (1950).

This is a comment on the fact that if $F(x)$ has bounded variation over $1\leqq x<\infty$, then the integral

$$\int_1^\infty F(x)dx=\lim_{A\to\infty}\int_1^A f(x)dx$$

and the series $\sum_{n=1}^{\infty}F(n)$ are both convergent or both divergent; see Agnew and Boas [Amer. Math. Monthly **56**, 677–678 (1949)] and references given there. The above hypothesis on $F(x)$ does not imply that $nF(n)\to 0$. In fact it is shown very simply that if $G(n)>0$ and $G(n)\to 0$, then there is a positive function $F(x)$ of bounded variation over $1\leqq x<\infty$ such that $\limsup_{n\to\infty} F(n)/G(n)=1$. The author could easily have varied his construction to make $\limsup F(n)/G(n)=\infty$. Taking $G(n)=1/n$, the author's construction gives a convergent series $\sum F(n)$ for which $\sum 2^kF(2^k)$ diverges and the condensation test for convergence is therefore inapplicable. *R. P. Agnew.*

Tsuchikura, Tamotsu. On some divergence problems. Tôhoku Math. J. (2) **2**, 30–39 (1950).

Examples and generalizations are given for known theorems of three types. A sequence p_n such that $0<p_1\leqq p_2\leqq\cdots$, $p_n\to\infty$, and $p_n/(p_1+\cdots+p_n)\to 0$ is constructed such that, for the Rademacher functions $r_n(x)$,

$$\limsup_{n\to\infty}\left|\frac{p_1r_1(x)+p_2r_2(x)+\cdots+p_nr_n(x)}{p_1+p_2+\cdots+p_n}\right|=1$$

almost everywhere. If $\sum a_k(x)$ is a series of measurable functions such that $\limsup_{n\to\infty}|\sum_{k=1}^{n}a_k(x)|=\infty$ for each x in a set E, then there is a sequence λ_k such that $\lambda_1\geqq\lambda_2\geqq\lambda_3\geqq\cdots$, $\lambda_k\to 0$, and $\limsup_{n\to\infty}|\sum_{k=1}^{n}\lambda_ka_k(x)|=\infty$ almost everywhere in E. If $\sum a_k(x)$ is a series of measurable functions such that $\limsup_{n\to\infty}|\sum_{k=1}^{n}a_k(x)|=\infty$ for each x in a set E, then there is an increasing sequence k_p of indices such that $k_{p+1}-k_p\to\infty$ and the subseries $a_{k_1}(x)+a_{k_2}(x)+\cdots$ has partial sums $t_n(x)$ for which $\limsup|t_n(x)|=0$ almost everywhere in E. The last theorem is extended to cover the case in which E is the union of subsets over each of which the series $\sum a_k(x)$ has a subseries having infinite oscillation. *R. P. Agnew* (Ithaca, N. Y.).

Lelong, Pierre. On a problem of M. A. Zorn. Proc. Amer. Math. Soc. 2 (11–19) (1951).

M. A. Zorn [Bull. Amer. Math. Soc. 53, 791–792 (1947); these Rev. 9, 139] has solved affirmatively the following problem that had been posed by the reviewer. If a formal power series $P(x_1, \cdots, x_k) = \sum a_{n_1 \cdots n_k} x_1^{n_1} \cdots x_k^{n_k}$ is such that for each set of constants $a_1, \cdots, a_k$ the one-variable power series $P(a_1 t, \cdots, a_k t)$ has a domain of absolute convergence, then the original k-variable series has one also. He proved this for complex constants $a_1, \cdots, a_k$, and Rimhak Ree [ibid. 55, 575–576 (1949); these Rev. 11, 25] proved it afterwards even for real constants.

Now, for $k=2$, only the ratio $\beta = a_2/a_1$ counts, and in the case of a complex β the author wants to determine point sets $E(\beta)$ in the β-plane having the property that whenever the power series $P(x, \beta x)$ has a set of convergence for each $\beta \in E(\beta)$, the series $P(x, y)$ has one also; and he finds the surprising answer that a necessary and sufficient condition for this to happen is that $E(\beta)$ shall not be a subset of a set of potential capacity zero. In the proof the theory of subharmonic functions (or, rather, of what we have termed Hartogs functions) is used extensively. *S. Bochner.*

Karamata, J. Complément à un théorème de M. Hadwiger. Comment. Math. Helv. 25, 64–70 (1951).

Let the series in $F(t) = \sum a_n t^n$ be convergent when $0 < t < 1$ and let $s_n = a_1 + a_2 + \cdots + a_n$. The two conditions $F(t) < M$, $0 < t < 1$, and $n a_n \geqq -W, n = 1, 2, 3, \cdots$, imply the condition $s_n \leqq W \log \log n + O(1)$. Moreover the function $\log \log n$ cannot be replaced by one which becomes infinite less rapidly. This shows that the condition $n a_n \geqq -W$ does not, as the condition $|n a_n| < M$ does, imply existence of a sequence t_n converging to 1 such that $\limsup |F(t_n) - s_n|$ is finite. Of several recent studies of this subject, one by Delange [Ann. Sci. École Norm. Sup. (3) 67, 99–160 (1950); these Rev. 12, 253] should be mentioned. *R. P. Agnew.*

Hamilton, Hugh J. Vector subseries of maximum modulus. Proc. Amer. Math. Soc. 2, 87–92 (1951).

The author extends to Euclidean n-space E_n, $n \geqq 2$, and to E_∞ his previous results for E_2 [Bull. Amer. Math. Soc. 44, 298–304 (1938)]. He shows (theorem 1) that if $\sum c_k$ is any absolutely convergent series of elements in a Banach space then it has a subseries $\sum c_j^*$ of maximum modulus. Write $r(c) = |\sum c_j^*| / \sum |c_k|$. Then (theorems 4 and 7) in E_n we have (*) $r(c) > \rho = \Gamma(\frac{1}{2}n) / \{2\pi^{\frac{1}{2}} \Gamma(\frac{1}{2}n + \frac{1}{2})\}$, this being the best possible value for ρ, and similarly in E_∞ with $\rho = 0$; this is sharper than A. E. Mayer's result $r(c) \geqq \rho$ [Comment. Math. Helv. 10, 288–301 (1938)]. For series that are not absolutely convergent (*) is replaced (theorem 9) by $\limsup_{N \to \infty} r(c^{(N)}) \geqq \rho$, where $c^{(N)}$ is the sequence $c_1, c_2, \cdots, c_N, 0, 0, \cdots$. *H. P. Mulholland* (Bath).

***Cooke, Richard G. Infinite Matrices and Sequence Spaces.** Macmillan & Co., Ltd., London, 1950. xiii + 347 pp. $6.50.

This book is intended to cover the theory of infinite matrices and its application to the theory of summation, further the theory of spaces whose elements are sequences of numbers. The author deliberately avoids consideration of special matrices (such as Cesàro's, Nörlund's, etc.), using them occasionally as examples, and concentrates on matrices of most general type. Chapters 1–3 deal with infinite matrices, their reciprocals, and equations of the types $AX - XD = 0$, $AX - XA = I$, X being the unknown matrix. Chapter 4 gives the Toeplitz theorem and its variants for sequence-to-sequence and series-to-sequence transformations. Schur's theorem is given only in the weaker form (due to Steinhaus), but its proof is reproduced in chapter 5 to derive some other result. Existence of methods which sum a given sequence to a prescribed value, or map it into a prescribed sequence, or sum a given sequence and do not sum some others, is proved. In chapter 5, consistency and equivalence of methods are considered (mostly for commutative matrices), further "absolutely regular" methods, that is methods which sum the difference of s_n and s_{n+1} to zero for a certain class of sequences s_n. The chapter ends with a discussion of consistency of the transformations $A(Bs_n)$ and $(AB)s_n$ due to Agnew. Chapter 6 gives a complete discussion of core theorems. Apart from the basic theorem of Knopp most of these are due to Agnew. Chapter 7 deals with "inefficiency theorems," which assert that a matrix fails to sum some, or all, divergent sequences. Some of these assume the existence of a reciprocal with certain properties, others give inefficiency with respect to power series. Chapter 8 is devoted to the "efficiency problem." Summation of power series by methods of Mittag-Leffler type is discussed. The theorem of Borel-Okada is proved. Theorems on efficiency for bounded sequences follow, together with results of Hill on summation of sequences of 0's and 1's. Chapter 9 gives an exposition of the theory of the Hilbert space of sequences (including theorems of Hilbert and Hellinger-Toeplitz on bilinear forms; the notion of an operator is not introduced). Chapter 10 deals (for the first time in any book) with Köthe-Toeplitz spaces, and contains also a short section on Banach spaces. Each chapter is followed by several pages of exercises.

The book is clearly written and is exhaustive in several interesting topics. As a book on general methods of summation, it is a welcome addition to Hardy's Divergent series [Oxford, 1949; these Rev. 11, 25], with which it does not overlap. Leaving aside its obvious strong features, we mention some of the defects of the book. The reviewer has the impression that not all topics treated are really important and that some of the most interesting and important results in the field are not discussed in the book. One particularly misses: (a) Theorems on consistency and equivalence of matrix methods obtained by Mazur, Banach, and Mazur-Orlicz (and later by Broudno, Hill, Darevsky, Zeller and others), which essentially depend on the topology of a sequence space; therefore, chapters 3–8 on linear methods and 9–10 on sequence spaces remain unconnected. (b) Tauberian theorems; besides some elementary Mercerian theorems and theorems on Tauberian conditions, one would especially welcome Karamata's general treatment of Tauberian theorems [Karamata, Sur les théorèmes inverses des procédés de sommabilité, Actualités Sci. Ind., no. 450, Hermann, Paris, 1937]. Among other missing topics, not contained in Hardy's book, are absolute summability, multiple sequences, theorems on nonequivalence of certain methods to matrix methods, transformations depending on integrals, etc. It is to be hoped that some of these questions will be treated in the second volume, planned by the author. The terminology of the book is different from the usual. Instead of "regular matrix," "T-matrix" is used, and instead of "equivalent," "mutually consistent." To the reviewer the terms "distinct diagonal matrix" for a diagonal matrix with distinct elements, "right value" for Abel's limit or "γ-matrix" for a regular series-to-sequence transformation seem unhappily chosen. Finally, not all proofs are the

best possible (for instance, (9.5, V) could be deduced at once from (9.4, VI), corollary 2). *G. G. Lorentz.*

Garreau, G. A. Absolute equivalence of general and row-finite T-matrices. Nederl. Akad. Wetensch. Proc. Ser. A. 54 = Indagationes Math. 13, 31–34 (1951).

Let $A=(a_{nk})$ be a regular Toeplitz matrix; if there exists a null sequence $\{\epsilon_n\}$ with the property that for each n there exists an integer k_n such that $|a_{nk}|<\epsilon^{k-k_n}$ when $k>k_n$, let B be the row-finite matrix for which $b_{nk}=a_{nk}$ when $k\leqq 2k_n$ and $b_{nk}=0$ when $k>2k_n$. The author shows that if $s_n=z^n$ $(n=0, 1, \cdots)$, where z is any complex number, the transforms As and Bs differ by a null sequence. Applying his result to the Lindelöf matrix and the Malmquist matrix [see the book reviewed above, p. 182], and using a theorem of Okada [Math. Z. 23, 62–71 (1925)], he obtains row-finite regular Toeplitz matrices that evaluate to $f(z)$ the sequence of partial sums of the Taylor series of every function $f(z)$, at each point in the principal star-domain of $f(z)$.

G. Piranian (Ann Arbor, Mich.).

Meyer-König, W. Beziehungen zwischen einigen Matrizen der Limitierungstheorie. Math. Z. 53, 450–453 (1951).

When $0<\alpha<1$ and $k=0, 1, 2, \cdots$, let matrices $T_\alpha^{(k)}$ and $S_\alpha^{(k)}$ be defined by

$$T_\alpha^{(k)}=(1-\alpha)^{n+k+1}\binom{n+k}{n+\nu}\alpha^{\nu-n},$$
$$S_\alpha^{(k)}=(1-\alpha)^{n+k+1}\binom{n+k+\nu}{\nu}\alpha^{\nu},$$

with row index $\nu=0, 1, 2, \cdots$ and column index $n=0, 1, 2, \cdots$. Let C_k and E_p denote the Cesàro and Euler-Knopp matrices of orders k and p respectively. Let $T_\alpha=T_\alpha^{(0)}$ and $S_\alpha=S_\alpha^{(0)}$; these matrices define circle methods of summability. Then $C_kT_\alpha=T_\alpha^{(k)}C_k$, $C_kS_\alpha=S_\alpha^{(k)}C_k$, and $E_pT_\alpha^{(k)}=S_\alpha^{(k)}$ when $\alpha=2^{-p}$. The matrices $T_\alpha^{(k)}$ and $S_\alpha^{(k)}$ are such simple modifications of T_α and S_α that the elegant commutation formulas facilitate study of relations among the matrices T_α, S_α, C_r, and E_p. For example, consistency of T_α and C_k is an obvious consequence of the formula $C_kT_\alpha=T_\alpha^{(k)}C_k$.

R. P. Agnew (Ithaca, N. Y.).

Hanai, Sitiro. On the methods of summation of infinite series. Tôhoku Math. J. (2) 2, 64–67 (1950).

Let a_{nk} be a regular matrix and let η_n be a convergent sequence. If to each $\epsilon>0$ corresponds a sequence x_k such that $|\sum_{k=1}^{\infty}a_{nk}x_k-\eta_n|<\epsilon$, $n=1, 2, \cdots$, then the conditions $\sum_{n=1}^{\infty}|\alpha_n|<\infty$ and $\sum_{n=1}^{\infty}\alpha_na_{nk}=0$, $k=1, 2, 3, \cdots$ imply that $\sum_{n=1}^{\infty}\alpha_n\eta_n=0$. This is a converse of a theorem of Banach [Théorie des opérations linéaires, Warsaw, 1932, p. 91].

R. P. Agnew (Ithaca, N. Y.).

Teghem, J. Sur des transformations de séries. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 36, 730–741 (1950).

Let α be a complex constant $(0\neq\alpha\neq 1)$, let $\sum a_n$ be any series, and let $p_n(\alpha)=(1-\alpha)^n\sum_{\nu=n}^{\infty}\binom{\nu}{n}\alpha^{\nu-n}a_\nu$; if the series defining the quantities $p_n(\alpha)$ converge, the series $\sum_0^\infty p_n(\alpha)$ is the Taylor transform $P(\alpha)$ of the series $\sum a_n$; the Taylor transformation is regular provided α is real and $0<\alpha<1$. Vermes [Amer. J. Math. 71, 541–562 (1949); these Rev. 10, 699] has shown that if the Taylor transform $P'(\alpha)$ of the series $u_1+u_2+\cdots$ converges to u, the Taylor transform $P(\alpha)$ of the series $u_0+u_1+u_2+\cdots$ converges to $u+u_0$, and that the converse holds whenever $\Re\alpha>\frac{1}{2}$. The author shows that if $P(\alpha)$ converges, $P'(\alpha)$ converges if and only if

$$\lim_{n\to\infty}\sum_{\nu=0}^{n}p_\nu(\alpha)(1-1/\alpha)^{n-\nu}=0$$

(the latter limit cannot exist and be finite without having the value zero). The result of Vermes follows from this by virtue of Merten's theorem.

The author defines the transform $\Pi(\alpha)$ of $\sum a_n$ as the series $\sum\pi_n(\alpha)$, where

$$\pi_n(\alpha)=(1-\alpha)^n\sum_{\nu=0}^{\infty}\binom{\nu+n}{\nu}^{\nu+1}a_\nu;$$

the transformation is closely related to the transformation S_α of Meyer-König [Math. Z. 52, 257–304 (1949); these Rev. 11, 242]. While in the case of the Taylor transforms the adjunction of a term to the series is always permissible, in the case of the transforms $\Pi(\alpha)$ the omission of a term is always permissible; the adjunction of a term is always permissible if $|1-\alpha|<1$. If $|1-\alpha|>1$, there exists precisely one number u_0 which may be adjoined to the series $u_1+u_2+\cdots$ as initial term without destroying convergence of its transform $\Pi(\alpha)$. If $|1-\alpha|=1$, there exists at most one such number u_0; a sufficient condition for its existence is the absolute convergence of $\Pi(\alpha)$. *G. Piranian.*

Garten, Viktor. Über eine Erweiterung der Sätze von G. Frobenius und O. Hölder in der Limitierungstheorie. Math. Nachr. 5, 129–134 (1951).

Generalizing integral transformations of Hölder and Cesàro, iterated mean values of functions are defined as follows. Let $0=P_0<P_1\leqq P_2\leqq\cdots$. Let $s(t)$ be integrable over each interval $0<t<x$. Let $\sigma^{(0)}(t)=s(t)$ and

$$\sigma^{(k)}(x)=\frac{P_k}{x^{P_k}}\int_0^x t^{P_k-1}\sigma^{(k-1)}(t)\,dt,\quad k=1, 2, 3, \cdots.$$

If $s(t)$ is such that, for some k, $\sigma_k(x)\to s$ as $x\to\infty$, then the Laplace mean $L(y)=y\int_0^\infty e^{-ty}s(t)\,dt$ exists for each sufficiently small positive y and $L(y)\to s$ as $y\to 0+$. Constants β_k, γ_k, and δ_k are defined such that, whenever $s(t)$ is such that the numbers $s_k=\liminf_{x\to\infty}\sigma^{(k)}(x)$, $S_k=\limsup_{x\to\infty}\sigma^{(k)}(x)$, $l=\liminf_{y\to 0+}L(y)$, $L=\limsup_{y\to 0+}L(y)$ are finite, the inequalities

$$\beta_ks_k-\delta_kS_k\leqq l\leqq L\leqq\beta_kS_k-\delta_ks_k$$

and $L-l\leqq\gamma_k(S_k-s_k)$ must hold. *R. P. Agnew.*

Kuttner, B. A note on some relations between methods of summability. J. London Math. Soc. 26, 111–116 (1951).

Let A and B denote two real transformations

$$A(x)=\sum_{n=0}^{\infty}a_n(x)s_n,\qquad B(x)=\sum_{n=0}^{\infty}b_n(x)s_n$$

of real sequences s_n. If the series for $B(x)$ converges and $B(x)\to L$ as $x\to\infty$ for each sequence s_n for which the series for $A(x)$ converges and $A(x)\to L$, we say that $A\to_{(1)}B$. If $A\to_{(1)}B$ and $\lim_{x\to\infty}\liminf_{N\to\infty}\sum_{n=0}^{N}b_n(x)s_n=\infty$ whenever the same relation holds for A, we say that $A\to_{(1)}B$. Finally if $\liminf_{N\to\infty}\sum_{n=0}^{N}b_n(x)s_n\geqq 0$, $0\leqq x<\infty$, whenever the same relation holds for A, we say that $A\to_{(1)}B$. Suppose $A\to_{(1)}B$. It appears from results involving Riesz, Cesàro, and other methods that the propositions $A\to_{(1)}B$ and $A\to_{(1)}B$ are, roughly speaking, much the same in the sense that, for most of the methods in common use, they are both true or both false. Instead of deducing results for the "$A\to_{(1)}B$ problem" from those for the "$A\to_{(1)}B$ problem" (or vice versa) by special arguments for each pair A, B, it would be desirable to have general results concerning these relations. Some pertinent examples and theorems are given.

R. P. Agnew (Ithaca, N. Y.).

Kuttner, B. Note on the "second theorem of consistency" for Riesz summability. J. London Math. Soc. **26**, 104–111 (1951).

Let k be a positive integer. A function $\varphi(t)$ has property P_k if evaluability of a sequence by a Riesz method $R(\lambda_n, k)$ for which $\lambda_n \geqq 0$, λ_n is increasing, and $\lambda_n \to \infty$ as $n \to \infty$ always implies evaluability of the sequence by the Riesz method $R(\varphi(\lambda_n), k)$. Let $\varphi(t)$ be a nonnegative increasing function, defined for $t \geqq 0$, such that $\varphi(t) \to \infty$ as $t \to \infty$. Suppose that, for t sufficiently great, $\varphi(t)$ is a $(k+1)$th indefinite integral of a function $\varphi^{(k+1)}(t)$. Then $\varphi(t)$ has property P_k if and only if there is a constant A such that

$$\int_A^\infty t^k |\varphi^{(k+1)}(t)|\,dt = \theta[\varphi(w)].$$

This is a refinement of theorems initiated by Hardy [Proc. London Math. Soc. (2) **15**, 72–88 (1916)] which give precision to the idea that $R(\varphi(\lambda_n), k)$ includes (or, in old terminology, is consistent with) $R(\lambda_n, k)$ provided $\varphi(t)$ increases reasonably regularly and not much more rapidly than t.

R. P. Agnew (Ithaca, N. Y.).

Fourier Series and Generalizations, Integral Transforms

Mohanty, R. The absolute Cesàro summability of the successively derived allied series of a Fourier series. Proc. Edinburgh Math. Soc. (2) **8**, 163–176 (1950).

If $f(t) \in L(-\pi, \pi)$, necessary and sufficient conditions for the r times differentiated allied (conjugate) series to be summable $|C, \alpha|$, $\alpha > 1$, at $t = x$ are that (i) a polynomial $P(t)$ should exist such that the odd function

$$g(t) = \tfrac{1}{2}r!\,t^{-r}\{f(x+t) + (-1)^{r+1} f(x-t) - P(t)\}$$

is integrable in the (CL) sense in $(0, \pi)$, (ii) the allied series of $g(t)$ should be summable $|C, \alpha+r|$ at $t=0$; the sum of the rth derived series is then $(2/\pi)\int_0^\infty (g(t)/t)\,dt$, where this integral exists in the $|CL|$ sense at the origin and is convergent. The (CL) (or $|CL|$) integral of a function over $(0, a)$ is the Cesàro (or absolute Cesàro) limit of its Lebesgue integral over (ϵ, a) as $\epsilon \to +0$. Analogous results have been given by the reviewer for the (C) summability of derived Fourier series [Quart. J. Math., Oxford Ser. (1) **10**, 67–74 (1939)], for the (C) summability of successively derived Fourier series [Proc. London Math. Soc. (2) **46**, 270–289 (1940); these Rev. **1**, 329], for the $|C|$ summability of derived Fourier series [Quart. J. Math., Oxford Ser. (1) **12**, 15–25 (1941); these Rev. **2**, 361], and for the (C) summability of successively derived allied series [Proc. London Math. Soc. (2) **49**, 63–76 (1945); these Rev. **7**, 154]. The author's proof is closely analogous to that in the last cited article.

L. S. Bosanquet (London).

Matsuyama, Noboru. Notes on Fourier analysis (XXV): On the $|C|$-summability of the Fourier Series. Tôhoku Math. J. (2) **2**, 51–56 (1950).

The author discusses the generalization of results concerning the absolute convergence of a Fourier series due to Bernstein [C. R. Acad. Sci. Paris **199**, 397–400 (1934)] and Szász [Trans. Amer. Math. Soc. **42**, 366–395 (1937)]. With weaker hypotheses his main result claims to establish summability $|C, \delta|$ almost everywhere. The reviewer however finds himself unable to reconstruct one of the author's estimates [notably the first of those involving $\delta_n(x)$ on p. 55].

L. S. Bosanquet (London).

Jurkat, W. Zur Konvergenztheorie der Fourier-Reihen. Math. Z. **53**, 309–339 (1950).

It was shown by F. T. Wang [Proc. London Math. Soc. (2) **47**, 308–325 (1942); these Rev. **4**, 37] that a necessary and sufficient condition for the summability of a Fourier series at a point in the Lebesgue set by Riesz means (R, ω^k, k), $k \geqq 1$, is $\int_{1/\omega}^{\pi/\omega} \varphi(t) t^{-1} \sin \omega t\,dt = o(1)$.

$$\varphi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - s\},$$

where $\mu(\omega)$ is a smoothly increasing function of a certain range. The range includes functions such as $\omega/\log \omega$, ω^α $(0 < \alpha < 1)$, $(\log \omega)^\alpha$ $(\alpha \geqq 1)$, the strength of the corresponding methods lying between convergence (with $\mu = \omega$) and (C, k) summability (with $\mu = \log \omega$). For $\mu = \omega^\alpha$ Wang was able to replace the condition $k \geqq 1$ by $k > 0$. Here the author extends the result for general μ to $k > 0$, and obtains an analogous result for the conjugate (allied) series. By imposing a continuity condition on the function, such as $\varphi(t) = o(1/\log t^{-1})$, together with an appropriate Tauberian condition on the coefficients, Wang [loc. cit.; Sci. Rep. Tôhoku Imp. Univ., Ser. 1. **24**, 697–700 (1936)] was able to proceed to convergence via Riesz summability, thus obtaining a new proof of a theorem of Hardy and Littlewood [J. London Math. Soc. **7**, 252–256 (1932); Ann. Scuola Norm. Super. Pisa (2) **3**, 43–62 (1934)]. Here the author obtains further results of the same type, in which the continuity and Tauberian conditions are governed by general functions.

L. S. Bosanquet (London).

Sunouchi, Gen-ichirô. Notes on Fourier analysis. XXXIX. Tôhoku Math. J. (2) **3**, 71–88 (1951).

This paper consists of several notes treating a number of disconnected topics. (1) Let $f(x)$ be of period 2π, integrable L, and with constant term of the Fourier series zero. Let $F(x)$ be a (periodic) primitive of f, and

$$\mu_r(\theta) = \left\{\int_0^\pi |F(\theta+t) + F(\theta-t) - 2F(\theta)|^r t^{-r-1}\,dt\right\}^{1/r}, \quad 1 < r < \infty.$$

Then, for $1 < p \leqq 2 \leqq q < \infty$, we have

$$A_{q,r}\|\mu_q\|_r \leqq \|f\|_r \leqq B_{p,r}\|\mu_p\|_r,$$

where $\|f\|_r = (\int_0^{2\pi} |f|^r dx)^{1/r}$ [see also Marcinkiewicz, Ann. Soc. Polon. Math. **17**, 42–50 (1938); Zygmund, Trans. Amer. Math. Soc. **55**, 170–204 (1944); these Rev. **5**, 230]. (2) Let $\Psi_x(t) = \{f(x+t) - f(x-t)\}/2 \sin \frac{1}{2}t$ be integrable (in the Cauchy sense) near $t = 0$. Suppose that the Fourier series of $\Psi_x(t)$ has partial sums $o(\log n)$ at $t = 0$ and that it is summable $(R, \log n, 1)$ there. Then the termwise differentiated Fourier series of f is summable $(R, \log n, 1)$ at the point x. Extension to summability $(R, \log n, k)$. (3) If $f \sim \sum_{-\infty}^{\infty} c_k e^{ikx}$ is of the (Hardy) class H, then $\sum c_k k^{-\epsilon} e^{ikx}$, $0 < \epsilon < 1$, is almost everywhere summable $(C, -\epsilon)$. (4) The theorem:

$$\int_0^{2\pi} \max_n \{|s_n(x)|^2/\log n\}\,dx \leqq A \int_0^{2\pi} |f|^2 dx$$

[Hardy and Littlewood, Proc. Cambridge Philos. Soc. **40**, 101–107 (1944); these Rev. **6**, 47] is extended to Fourier integrals and to double Fourier series. (5) In one-side localization theorems (see the reviewer's Trigonometrical Series [Warszawa-Lwow, 1935], p. 273) the condition $o(1/n)$ for the coefficients a_n, b_n, can be replaced by $\sum_1^n k(a_k^2 + b_k^2)^{\frac{1}{2}} = o(n)$.

A. Zygmund (Chicago, Ill.).

Sunouchi, Gen-ichiro. Notes on Fourier analysis. XXXVI. On certain applications of Wiener's Tauberian theorems. Tôhoku Math. J. (2) 1, 303–312 (1950).

By substituting kernels involving Bessel functions into general Tauberian and quasi-Tauberian theorems of Wiener [Ann. of Math. (2) 33, 1–100 (1932)] and the author [same vol., 167–185 (1950); these Rev. 12, 174] results are obtained which include a theorem of Cheng [Ann. of Math. (2) 50, 763–776 (1949); these Rev. 11, 347] and the quasi-Tauberians of Chandrasekharan [Proc. London Math. Soc. (2) 50, 210–222, 223–229 (1948); these Rev. 10, 113] concerning the summability of multiple Fourier series by spherical Riesz-Bochner means. *L. S. Bosanquet.*

Berkovitz, Leonard D. Circular summation and localization of double trigonometric series. Trans. Amer. Math. Soc. 70, 323–344 (1951).

A double trigonometrical series is denoted in the form $T=\sum a_M e^{iMX}$, where M is a pair of integers m, n which run from $-\infty$ to ∞ and X is a pair of real numbers x, y in $(0, 2\pi)$, and where $MX=mx+ny$; and the Bochner sum (C, α) is denoted by $\sigma_R^\alpha(x, y)=\sum_{|M|\leq R} a_M e^{iMX}(1-|M|/R)^\alpha$ where $\alpha>0$ and $|M|=(m^2+n^2)^{\frac{1}{2}}$. The series T is said to be Bochner summable (C, α) (or circularly summable (C, α)) to $L(x, y)$ when $\sigma_R^\alpha(x, y)$ tends to $L(x, y)$ as $R\to\infty$. The author develops the theory of formal multiplication and Riemann's theory of localization of double trigonometrical series with respect to this summation. Among other things, he proves the following. (1) Let T be a double trigonometrical series with coefficients $a_M=o(|M|^\gamma)$, $\gamma\geqq -1$. Then the series

$$\frac{a_{00}(x+y)^{2k}}{2^k\cdot(2k)!}+\sum_{|M|\neq 0}\frac{(-1)^k a_M}{|M|^{2k}}e^{iMX}$$

converges uniformly to a function $F(x, y)$ with $k=[\frac{1}{2}\gamma]+2$. (2) If T and T' are two double trigonometrical series with coefficients $o(|M|^\gamma)$, $\gamma\geqq -1$, and if the functions F and F' associated with each of the series in the above sense are equal in a closed domain $\mathfrak{R}$ in the rectangle $[0\leqq x\leqq 2\pi, 0\leqq y\leqq 2\pi]$, then in every closed domain completely contained in $\mathfrak{R}$, the series $T-T'$ is uniformly Bochner summable $(C, \gamma+1)$ to zero. *S. Izumi* (Tokyo).

Wang, Shou-jen. On strong summability of multiple Fourier series. Acad. Sinica Science Record 2, 350–358 (1949).

It was proved by Hardy and Littlewood [Fund. Math. 25, 162–189 (1935)] that if $\int_0^t|f(x+u)+f(x-u)|\,du=o(t)$ as $t\to+0$, then $\sum_{\nu=0}^n\{s_\nu\}^2=o(n\log n)$, where $f(x)\in L$ and s_n is the partial sum of its Fourier series. This is the case $p=0$, $k=1$ of the following theorem given by the author. If $\int_0^t|f_p(u)|\,du=o(t)$ as $t\to+0$, $p\geqq 0$, then

$$\int_0^R\{S^\delta(r)\}^2=o(R\log R),$$

$\delta=p+\frac{1}{2}(k-1)$, where $f(x_1, \cdots, x_k)\in L$, $f_p(t)$ is the spherical mean of f about $(x_1, \cdots, x_k)$, and $S^\delta(r)$ is the spherical mean of the multiple Fourier series of f at $(x_1, \cdots, x_k)$. The proof makes use of certain lemmas of Bochner and Chandrasekharan who had similarly extended another theorem on the strong summability of Fourier series [Bochner, Trans. Amer. Math. Soc. 40, 175–207 (1936); Bochner and Chandrasekharan, Ann. of Math. (2) 49, 966–978 (1948); these Rev. 10, 248]. *L. S. Bosanquet* (London).

Lozinskiĭ, S. M. On the convergence of double Fourier series. Doklady Akad. Nauk SSSR (N.S.) 72, 841–843 (1950). (Russian)

The author gives some sufficient conditions for the convergence of Fourier series of functions of two variables. Denote by $\bar{C}$ the class of continuous functions of period 2π with the norm

$$\|f\|_{\bar{C}}=\max_{x,y}|f(x, y)|.$$

Let Ω be a class of functions $\omega(u)$ which satisfy (1) $\omega(u)$ continuous for $0\leqq u<\infty$, (2) $0<\omega(u')\leqq\omega(u'')$ for $0<u'\leqq u''$, (3) $\omega(u_1+u_2)\leqq\omega(u_1)+\omega(u_2)$, (4) $\omega(0)=0$, (5) $u/\omega(u)\to 0$ as $u\to 0$. If $\omega_1\in\Omega$ and $\omega_2\in\Omega$, let $\bar{C}_{\omega_1,\omega_2}$ be the sub-class of $\bar{C}$ for which $|f(x+\alpha, y+\beta)-f(x, y)|\leqq A\omega_1(|\alpha|)+B\omega_2(|\beta|)$, where A and B may depend on f but are independent of x, y, α, and β.

Let Π be a set of pairs (n, m) of positive integers with the property that corresponding to any $N>0$, we can find a pair $(n, m)\in\Pi$ such that $n>N$, $m>N$. Let $s_{n,m}$ denote the partial sum of the Fourier series of a function $f(x, y)$ and write

$$\overline{\lim_{\Pi}}\,s_{n,m}=\lim_{N\to\infty}\ \inf_{n\geqq N,\,m\geqq N,\,(n,m)\in\Pi}s_{n,m}$$

and let $\overline{\lim}_{\Pi}s_{n,m}$ denote the corresponding expression with inf replaced by sup. When these two limits are equal denote their common value by $\lim_{\Pi}$.

The author proves that if the above conditions are satisfied and if in addition

$$\lim_{\Pi}\min[\omega_1(1/n), \omega_2(1/m)]\log n\log m=0,$$

$$\lim_{\Pi}\omega_1(1/n)\log n=\lim_{\Pi}\omega_2(1/m)\log m=0,$$

then for all f of $\bar{C}_{\omega_1,\omega_2}$ we have $\lim_{\Pi}\|f-s_{n,m}(f)\|_{\bar{C}}=0$. The remaining theorems are special cases of this result. *A. C. Offord* (London).

Rudin, Walter. Uniqueness theory for Hermite series. Trans. Amer. Math. Soc. 70, 387–403 (1951).

The paper is concerned with the problem of finding sufficient conditions under which a given series of Hermite functions $\phi_n(x)$ is the Hermite series of a function $f(x)$ of class H. Here $H=\bigcap_{p\geqq 0}H_p$ and $f(x)\in H_p$ if $f(x)\in L$ on every finite interval and $x^p e^{-x^2/2}f(x)\in L(-\infty, \infty)$. The discussion is based on the use of two inverse Hermite operators Λ and Ω corresponding to the differential and integral operators used in the Riemann theory of trigonometric series. Here $\Lambda\phi_n(x)=-(2n+2)\phi_n(x)$ and $\Omega\phi_n(x)=-(2n+2)^{-1}\phi_n(x)$. If $F''(x)$ exists then $\Lambda F(x)=F''(x)-(x^2+1)F(x)$; in general Λ, Λ^*, Λ_* are the lim, lim sup, lim inf for $h\to 0$ of $2h^{-2}\Delta_h F(x)$ where $\Delta_h F(x)=y(x; F, h)-F(x)$ and $y(t; F, h)$ is the solution of $y''-(t^2+1)y=0$ for which $y(x\pm h)=F(x\pm h)$. If $f\in H_p$, p fixed $\geqq 0$, Ωf is the (unique) solution of $y''-(x^2+1)y=f$ which is in H_p. Here $\Lambda\Omega f(x)=f(x)$ for almost all x and $\Lambda^*\Omega f(x)\leqq f(x)$ if f is upper semi-continuous at x. Finally, $f^*(x)=\limsup_{r\to 1}f(x, r)$ when $f(x, r)=\sum_0^\infty a_n\phi_n(x)r^n$ is supposed convergent for $0\leqq r<1$, and $f_*(x)$ is the corresponding lim inf. The analysis is based on a theorem 1 ensuring the existence of $\Lambda F(x)$ almost everywhere provided (a) $F\in H_p$, $p\geqq 0$, and F is continuous, (b) $\Lambda^* F(x)>-\infty$ and $\Lambda_* F(x)<+\infty$ except possibly on countable sets E_1 and E_2, (c) $\limsup_{h\to 0}h^{-1}\Delta_h F(x)\geqq 0$ on E_1 and $\liminf_{h\to 0}h^{-1}\Delta_h F(x)\leqq 0$ on E_2, (d) there exists a function $y\in H_p$ such that $y(x)\leqq\Lambda^* F(x)$ for all x. These conditions also ensure $\Lambda F\in H_p$ and $F(x)=\Omega\Lambda F(x)$ for all x. [Cf. the author's dissertation [Trans.

Amer. Math. Soc. 68, 278–286 (1950); these Rev. 11, 663] where a corresponding theorem is proved for the generalized Laplacian.] The main results are now: (2) If $f \in H$, then $f(x) \sim \sum_0^\infty a_n \phi_n(x)$ if and only if $\Omega f(x) \sim -\sum_0^\infty a_n (2n+2)^{-1} \phi_n(x)$. (3) Suppose a continuous function $F(x)$ satisfies (b), (c), and (d) and $F(x) \sim -\sum_0^\infty a_n (2n+2)^{-1} \phi_n(x)$. Then $\Lambda F(x) \sim \sum_0^\infty a_n \phi_n(x)$. Counterexamples show that the theorem may fail if $F(x)$ is not continuous or (c) is violated. (4) An analogue of Rajchman's inequalities for trigonometric series is proved. (5) A condition is found in order that a series of Hermite functions be Abel-Poisson summable almost everywhere and be the Hermite series of its Abel-Poisson sum. (6) The conclusion under (5) holds in particular if $a = o(n^{\frac{1}{2}})$ and if there exists a $y \in H$ such that

$$-\infty < y(x) \leqq f_*(x) \leqq f^*(x) < \infty$$

for all x. Here $a_n = O(n^{\frac{1}{2}})$ is not sufficient. (7) If the terms of a series of Hermite functions tend to zero on a set of positive measure, then $a_n = o(n^{\frac{1}{2}})$. (8) A series of Hermite functions, converging for all x to a finite function $f(x)$ in H, is the Hermite series of its sum. [To the references should be added: A. Gonzáles Domínguez, Ciencia y Técnica 42, 283–329 (1941); these Rev. 12, 330].

E. Hille (New Haven, Conn.).

Mitchell, Josephine. Convergence and $(C, 1, 1)$ summability of double orthogonal series. Duke Math. J. 18, 211–219 (1951).

Let $\{\phi_m\}$, $\{\psi_n\}$ be complete orthonormal systems of functions defined over the sets E_1 and E_2 respectively, where E_1, E_2 are each embedded in Euclidean space and of finite measure. The orthogonal development of $f \in L^2$ is (1) $\sum a_{mn} \phi_m \psi_n$ where $a_{mn} = \int_E \phi_m \psi_n f dA$, $E = E_1 \times E_2$. The author proves a theorem of which the simplest case shows that if the kernel and Lebesgue functions

$$K_{1n}(x, s) = \sum_{j=1}^{m} \phi_j(x) \phi_j(s), \quad K_{2n}(y, t) = \sum_{k=1}^{n} \psi_k(y) \psi_k(t),$$

$$L_{1m} = \int_{E_1} |K_{1m}(x, s)| dx, \quad L_{2n} = \int_{E_2} |K_{2n}(y, t)| dy$$

satisfy (i) K_{1m}, K_{2n} are nonnegative and (ii) L_{1m}, L_{2n} are bounded, then the series (1) is bounded almost everywhere (a.e.) and converges a.e. on E. A similar result is established for $(C, 1, 1)$ summability in terms of appropriate kernel and Lebesgue functions, and this last theorem is used to establish a condition sufficient for the convergence of a Fourier-Haar series a.e. on $E(0 \leqq x \leqq 2\pi, 0 \leqq y \leqq 1)$. *H. G. Eggleston.*

Bohr, Harald. On the definition of almost periodicity. J. Analyse Math. 1, 11–27 (1951). (English. Hebrew summary)

Call a sequence $\{t_\nu\}$ "discrete" if $\inf_{\nu \neq \mu} |t_\nu - t_\mu| > 0$. If $f(x)$ is continuous in $-\infty < x < \infty$, then for $\epsilon > 0$ denote by $E_f(\epsilon)$ the set of points t for which $|f(x+t) - f(x)| < \epsilon$ for all x, and take some "discrete" sequence in $E_f(\epsilon)$. For each $l > 0$ denote by $m(x_0, l)$ the number of elements of the latter sequence that are located in $x_0 \leqq x < l$. The original definition of almost periodicity demands that for any ϵ, for some $\{t_\nu\}$ there shall be $\min_{-\infty < x_0 < \infty} m(x_0, l) > 0$ for some l, but Følner [Mat. Tidsskr. B. 1944, 24–27; these Rev. 7, 60] reduced this condition to $\liminf M_l/l > 0$, where $M_l = \max_{-\infty < x_0 < \infty} m(x_0, l)$ (note the "max"). Now the author takes a positive function $\varphi(l)$ and he investigates the new condition $\liminf M_l/\varphi(l) > 0$ and he shows that whenever $\varphi(l)/l \to 0$, there is a function $f(x)$ which satisfies the new condition but is not almost periodic any longer. Thus Følner's condition cannot be weakened much further. *S. Bochner* (Princeton, N. J.).

Marčenko, V. A. Generalized almost periodic functions. Doklady Akad. Nauk SSSR (N.S.) 74, 893–895 (1950). (Russian)

The author obtains results closely related to those of Levitan [Mat. Sbornik N.S. 24(66), 321–346 (1949); these Rev. 11, 174]. His result is too lengthy to state here.

N. Levinson (Cambridge, Mass.).

Behrens, Ernst-August. Über Funktionen auf der Drehungsgruppe. Arch. Math. 2, 337–345 (1950).

An almost periodic (a. p.) function $f(x)$ on a group $\mathfrak{G}$ is one such that for $\epsilon > 0$ there is a finite partition of $\mathfrak{G}$ such that if x and y belong to the same partition set and if c and d are arbitrary, then $|f(cxd) - f(cyd)| < \epsilon$. Let $\vartheta_\nu = (D^\nu_{\mu\sigma}(t))$ be a system of inequivalent irreducible unitary representations of $\mathfrak{G}$, one representation from each class; let M_t be the von Neumann integral mean on $\mathfrak{G}$, $\chi_\nu(t)$ the character of $D^\nu_{\mu\sigma}(t)$, s_ν the degree of $D^\nu_{\mu\sigma}(t)$. For an a. p. $f(x)$ on $\mathfrak{G}$ one defines $S_n(x, f)$ to be

$$\sum_{\nu=1}^{n} s_\nu \Big(\sum_{\mu,\sigma} M_t\{f(t)\overline{D^\nu_{\mu\sigma}(t)}\} D^\nu_{\mu\sigma}(x) = M_t\{f(xt^{-1}) \sum_{\nu=1}^{n} s_\nu \chi_\nu(t)\},$$

the Dirichlet integral with kernel $K_n(t) = \sum_{\nu=1}^{n} s_\nu \chi_\nu(t)$. For the case where $\mathfrak{G}$ is the set of all 3×3 orthogonal matrices of determinant 1, one finds that the kernel can be expressed as a function of a single real variable $K_n(\alpha)$, $0 \leqq \alpha \leqq \pi$, the size of the angular rotation produced by the element of the group. The a. p. functions on $\mathfrak{G}$ coincide with the uniformly continuous functions. As a consequence of theorems of van der Waerden and von Neumann the representations of $\mathfrak{G}$ are infinitely differentiable. For each k [see van der Waerden, Die gruppentheoretische Methode der Quantenmechanik, Springer, Berlin, 1932] there is precisely one class of continuous irreducible representations of degree $s_k = 2k+1$, and the character

$$\chi_k(t) = \chi_k(\alpha) = \sum_{-k}^{k} e^{i\nu\alpha} = \sin\left(\tfrac{1}{2}(2k+1)\alpha\right) \cdot (\sin \tfrac{1}{2}\alpha)^{-1};$$

$$S_n(x, f) = (1/\pi) \int_0^\pi g(\alpha, x) K_n(\alpha) d\alpha,$$

where $g(\alpha, x)$ $(1/4\pi) \int_{-\pi/2}^{\pi/2} \int_{-\pi}^{\pi} f(xt^{-1}(\alpha, \beta, \gamma)) \cos\beta d\gamma d\beta$ and α, β, γ are the canonical group parameters of $\mathfrak{G}$. If we extend the definition of $g(\alpha, x)$ by $g(-\alpha, x) = g(\alpha, x) = g(\alpha + 2\pi, x)$, and let $s_n(g)$ and $a_n(g)$ represent the nth partial sum and nth term respectively of the Fourier series of $g(\alpha, x)$ relative to α, then $S_n(x, f) = \frac{1}{2} s_{n+1}(g) + \frac{1}{2} s_n(g) - (n+1) a_{n+1}(g)$. The function $f(t)$ is called regular at x if $\lim_{\alpha \to 0+} g(\alpha, x) = \gamma$ exists. The following theorems are obtained: (1) If $f(t)$ is regular at x, the H^1-summability of its Fourier series on $\mathfrak{G}$ at x is equivalent to the convergence of the Fourier series of $g(\alpha, x)$ on $(-\pi, \pi)$ at $\alpha = 0$. When the condition is fulfilled, the two series have the common limit γ. (2) If $f(t)$ is regular at x, then the convergence of its Fourier series on $\mathfrak{G}$ at x is equivalent to: $\lim_{n \to \infty} n a_n(g) = 0$. (3) For a given even periodic integrable function $h(\alpha)$ for which $\lim_{\alpha \to 0+} h(\alpha) = \eta$ exists, define $f(x)$ to be $h(\alpha(x^{-1}x_0))$, where $\alpha(t)$ is the rotation angle of t. Then the Fourier series of $h(\alpha)$ on $(-\pi, \pi)$ converges at $\alpha = 0$ if and only if the Fourier series of $f(x)$ on $\mathfrak{G}$ is H^1-summable at $x = x_0$. Hence, not every function which is regular at x has an H^1-summable Fourier series on $\mathfrak{G}$ at x, and the Riemann localisation theorem is shown, by example, to be false for functions on $\mathfrak{G}$. *B. Gelbaum.*

Mitra, S. C., and Sharma, A. On certain self-reciprocal functions. Ganita 1, 31–38 (1950).

This is an addition to two earlier papers by Mitra [Bull. Calcutta Math. Soc. 41, 1–5, 49–52 (1949); these Rev. 11, 350, 351] and gives further examples of series of Bessel and associated functions which are self-reciprocal in the Fourier sine or cosine transform. The authors also obtain an integral equation for a function $f(x)$ satisfying the condition that the series $f(x)-f(3x)-f(5x)+f(7x)+\cdots$ should be self-reciprocal in the Hankel transform of order ν.

M. C. Gray (Murray Hill, N. J.).

Agarwal, Ratan Prakash. On self-reciprocal functions involving two complex variables. Ganita 1, 17–25 (1950).

This paper extends the work on self-reciprocal functions presented in chapter IX of Titchmarsh's "Introduction to the Theory of Fourier Integrals" [Oxford, 1937] to two variables. The proofs are based on two theorems by I. S. Reed [Duke Math. J. 11, 565–572 (1944); these Rev. 6, 49] on double Mellin integrals. The principal result is the statement that the double Mellin transform $F(r, s)$ of a function which is self-reciprocal in the double Hankel transformation of orders μ, ν satisfies the functional equation

$$F(r,s)=2^{r+s-1}\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}r+\frac{1}{4})\Gamma(\frac{1}{2}\mu+\frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}r+\frac{3}{4})\Gamma(\frac{1}{2}\mu-\frac{1}{2}s+\frac{3}{4})}F(1-r,1-s).$$

This corresponds to Titchmarsh's equation (9.1.8). As in Titchmarsh's book, this functional relation is used to establish the connection between classes of self-reciprocal functions.

A. Erdélyi (Pasadena, Calif.).

***Voelker, Dietrich, und Doetsch, Gustav. Die zweidimensionale Laplace-Transformation. Eine Einführung in ihre Anwendung zur Lösung von Randwertproblemen nebst Tabellen von Korrespondenzen.** Verlag Birkhäuser, Basel, 1950. 259 pp. 39 Swiss francs; bound, 43 Swiss francs.

This treatise appears to be an outgrowth of the junior author's Freiburg dissertation [D. Voelker, Die zweidimensionale Laplace-Transformation und ihre Anwendung zur Lösung von Systemen partieller Differentialgleichungen, Würzburg, 1939]. It is divided into two parts. The first is devoted to the theory of the two-dimensional Laplace transformation and its use in solving boundary value problems. A brief review of the one-dimensional case is followed by the definition and (absolute) convergence theory of the two-dimensional transform and its behavior under the basic operations of linear substitutions, differentiation, and convolution, a total of 25 pages. The Lebesgue integral is used throughout and the authors make an eloquent plea for its use by the engineer and the applied mathematician, the class of readers to which this treatise is directed primarily. This section is followed by 115 pages devoted to boundary value problems for partial differential equations (first order, special second order, general second order, systems of two first order equations, and the heat equation for the plane). The boundary value problems are normally formulated for the first quadrant of the plane; the double Laplace transform translates the partial differential equation plus the boundary conditions into an ordinary linear algebraic equation from the solution of which the desired solution is obtained by inversion. The method appears to be one of great power and range. Much attention is devoted to the intricate consistency conditions for the boundary values and the various steps are carried out in detail and with great care so as to provide the reader interested in special applications with good reliable working models. The first part also contains a brief section on functional relations and series expansions obtainable by the double Laplace integral. There is also a one page bibliography. Part two, by D. Voelker alone, gives 106 pages of correspondences between functions of two variables and their double Laplace transforms. These tables are used extensively in the first part and are indispensible to those who want to apply the method to special problems.

E. Hille (New Haven, Conn.).

***Thomson, William Tyrrell. Laplace Transformation. Theory and Engineering Applications.** Prentice-Hall, Inc., New York, N. Y., 1950. ix+230 pp. $3.75.

This book consists of nine short chapters dealing with the following topics: introduction to and properties of the Laplace transformation, applications to dynamics and to structures, complex variables, partial differential equations, difference equations, closed-loop systems and analogies; the appendices include a table of transforms and a bibliography. Elementary theory is presented very briefly. Some proofs are quite heuristic, evidently presented as a way of making the results seem plausible rather than as a valid proof. Step and impulse functions are used frequently; continuity conditions, omitted in the theorems, are necessary for treatment of such functions. The physical applications include examples and problems not usually found in other books on operational mathematics. The presentation of some of the applications here is not complete enough to be clear or self-contained, but these parts of the book may be useful to the reader already familiar with the physical problem.

R. V. Churchill (Ann Arbor, Mich.).

Humbert, Pierre. Une nouvelle formule opératoire. C. R. Acad. Sci. Paris 232, 1397–1398 (1951).

Let $\phi(p)/p$ denote the Laplace transform of a function $f(t)$ and let $\lambda(p)$ denote the function $\log[p+(p^2+1)^{\frac{1}{2}}]$. The formula

$$(p^2+1)^{-\frac{1}{2}}\phi[\lambda(p)]/\lambda(p)=L\left\{\int_0^\infty J_s(t)f(s)ds\right\}$$

is derived formally, where L denotes the Laplace transformation and J_s is Bessel's function of the first kind.

R. V. Churchill (Ann Arbor, Mich.).

Charles, H. Sur l'application du calcul symbolique. I. Bull. Soc. Roy. Sci. Liège 19, 469–474 (1950).

Charles, Henri. Sur le calcul symbolique et ses applications. II. Bull. Soc. Roy. Sci. Liège 19, 546–549 (1950).

A few theorems on basic properties of Laplace transforms are stated as a result of arguments involving certain delta functions.

Reviewer's remarks: The author's theorem II of part II states that if a continuous function $F(t)$ with a sectionally continuous derivative $F'(t)$ has a transform then $F(t)$ is of exponential order as t tends to infinity. The function $F(t)=2te^{t^2}\cos e^{t^2}$ serves as a simple counterexample. This function is the derivative of the bounded function $\sin e^{t^2}$ and therefore the transform of $F(t)$ exists [theorem 1, page 6 of the reviewer's Modern Operational Mathematics . . . , McGraw-Hill, New York, 1944; these Rev. 5, 267]; but $F(t)$ is not of exponential order. The same function together with theorem 3, page 101 of G. Doetsch, Handbuch der Laplace-Transformation, vol. 1 [Birkhäuser,

Basel, 1950] serves as a counterexample of the author's theorem I of part II. Other theorems are not fully stated.

R. V. Churchill (Ann Arbor, Mich.).

Polynomials, Polynomial Approximations

Sz.-Nagy, Gyula. Kurzer Beweis eines Satzes über die obere Schranke der absoluten Beträge von mehreren Nullstellen eines Polynoms. Publ. Math. Debrecen 1, 251–253 (1950).

The author gives another proof of Montel's and Van Vleck's theorems that at least p zeros of the polynomial $f(z)=a_0+a_1z+\cdots+a_pz^p+\cdots+a_nz^n$ lie in each of the circles $|z|\leqq r_1$ and $|z|\leqq r_2$, where r_1 and r_2 denote respectively the positive zeros of the polynomials

$$F_1(z)=|a_p|z^p-\sum_1^p C(n-p+k,k)|a_{p-k}|z^{p-k}, \qquad a_0a_p\neq 0,$$

$$F_2(z)=|a_n|z^n-\sum_1^p C(n-p+k-1,k-1)|a_{p-k}|z^{p-k}, \quad a_0a_n\neq 0,$$

and where $C(n,k)$ denotes the binomial coefficient. His proof consists in showing that at most $n-p$ zeros of $g(z)=z^nf(1/z)$ lie in the circle $|z|<1/r_1$. This is done by setting $G(z)=g(z)/[(z-z_1)\cdots(z-z_{n-p})]$, where $z_1,\cdots,z_{n-p}$ are zeros of $g(z)$ in the circle $|z|<1/r_1$, and by establishing that $r_1^p|G(z)|>-F_1(r_1)=0$ for $|z|>1/r_1$. [Reviewer's note: This method of proof is similar to one due to Montel. Cf. Marden, Geometry of the Zeros . . . , Amer. Math. Soc., New York, 1949, pp. 113–115; these Rev. 11, 101.]

M. Marden (Milwaukee, Wis.).

Toscano, Letterio. Una classe di polinomi della matematica attuariale. Rivista Mat. Univ. Parma 1, 459–470 (1950).

The author studies the polynomials $G_n^{(\alpha)}$ generated by the expansion

$$\exp[\alpha t+x(1-e^t)]=\sum_{n=0}^{\infty}(n!)^{-1}t^nG_n^{(\alpha)}(x).$$

Starting with the (known) formulas

$$G_n^{(\alpha)}(x)=x^{-\alpha}e^x\delta^n\{x^\alpha e^{-x}\}=e^{-x\Delta}\alpha^n,$$

in which δ is the operator of differentiation with respect to $\log x$, and Δ is the operator of differencing with interval unity with respect to α, he obtains many properties, old and new, of the $G_n^{(\alpha)}(x)$. He investigates in particular the connection between the $G_n^{(\alpha)}$ and Laguerre polynomials, and the various formulas emerging from this connection. The reviewer should like to add to the list of references: E. Hilb [Math. Ann. 85, 89–98 (1922)] and K. Mahler [Rend. Circ. Mat. Palermo 54, 1–41 (1930)].

A. Erdélyi.

Straus, Ernst G. On the polynomials whose derivatives have integral values at the integers. Proc. Amer. Math. Soc. 2, 26–27 (1951).

It is proved that a necessary and sufficient condition for a polynomial $f(x)$ and all its derivatives to have integral values for all the integers is that $f(x)=\sum_{n=0}^{m}c_nf_n(x)$, where the coefficients are integers and

$$f_n(x)=x(x-1)\cdots(x-n+1)\prod p^{[n/p]}/n!,$$

the product ranging over all primes. A generalization is stated without proof, and several related unsolved problems are suggested.

A. W. Goodman (Lexington, Ky.).

Sz.-Nagy, Béla. Über die Konvergenz von Reihen orthogonaler Polynome. Math. Nachr. 4, 50–55 (1951).

The author sharpens a result of Alexits [Comment. Math. Helv. 16, 200–208 (1944); these Rev. 5, 262] by proving the following theorem: Let $\alpha(x)$ be a nondecreasing function on $[-1,1]$ taking on an infinite number of values. The set of discontinuities of $\alpha(x)$ is denoted by E and the set of points at which $\alpha(x)$ has a finite derivative by E'. Let $\{p_n(x)\}$ be the orthonormal set of polynomials associated with the distribution $d\alpha(x)$, and assume that there is an open interval $\Delta\subset[-1,1]$ such that the $p_n(x)$ are bounded in every closed subinterval δ of Δ. Then, if $\sum c_n^2\log n<\infty$, the series (*) $\sum c_np_n(x)$ converges everywhere in E and almost everywhere (with respect to the measure $d\alpha$) in $\Delta'E'$. Further, if $\alpha'(x)>0$ almost everywhere in Δ then (*) converges almost everywhere in the ordinary sense.

M. J. Gottlieb (Chicago, Ill.).

Fejér, Leopold. Beste Approximierbarkeit einer gegebenen Funktion durch ein Polynom gegebenen Grades, wenn das Polynom sonst beliebig oder wenn es noch einer interpolatorischen Beschränkung unterworfen ist. Math. Nachr. 4, 328–342 (1951).

The author gives the background and a short review of some of his results on interpolation obtained in the first third of the present century. He discusses interpolation on normal point groups, their relations to quadrature problems, and problems of best approximations. The points $-1\leqq x_1<x_2<\cdots<x_n\leqq 1$ are called by the author normal if all the points $X_k=x_k+\omega'(x_k)/\omega''(x_k)$, $k=1,2,\cdots,n$, are outside the interval $(-1,+1)$, where $\omega(x)=\prod_{i=1}^{n}(x-x_i)$. The author proves in detail the following theorem. Let $f(x)$ be a continuous function, $x_1,x_2,\cdots,x_n$ a normal point group. Denote by $T_{2n-1}(x)$ the polynomial of degree $2n-1$ or less for which $\max_{-1\leqq x\leqq 1}|f(x)-g_{2n-1}(x)|$ is a minimum, where $g_{2n-1}(x)$ runs through all polynomials of degree $2n-1$ or less. Denote by $F_{2n-1}(x)$ the unique polynomial of degree $2n-1$ or less for which $F_{2n-1}(x_k)=f(x_k)$, $F'_{2n-1}(x_k)=T'_{2n-1}(x_k)$, $k=1,2,\cdots,n$. Then

$$(1)\quad \max_{-1\leqq x\leqq 1}|T_{2n-1}(x)-f(x)|\leqq \max_{-1\leqq x\leqq 1}|F_{2n-1}(x)-f(x)| \leqq 2\max_{-1\leqq x\leqq 1}|T_{2n-1}(x)-f(x)|,$$

and in general the factor 2 can not be replaced by any smaller constant. The result (1) is surprising since it shows that among the class of polynomials $g_{2n-1}(x)$ of degree $2n-1$ or less which satisfy $g_{2n-1}(x_k)=f(x_k)$, $k=1,2,\cdots,n$, there is one which approximates almost as well to $f(x)$ as $T_{2n-1}(x)$.

P. Erdös (Aberdeen).

Berman, D. L. On the approximation of continuous functions by interpolating polynomials. Doklady Akad. Nauk SSSR (N.S.) 77, 173–176 (1951). (Russian)

We consider a triangular matrix of interpolation points $(x_k^{(n)})$ on $[-1,1]$ and the corresponding Lagrange interpolation polynomials $L_n(f,x)$ for a continuous $f(x)$; then we do not necessarily have $L_n(f,x)\to f(x)$, but S. Bernstein constructed (for certain interpolation points) other interpolation polynomials $A_n(f,x)$ of degree $m_n>n$ such that $A_n(f,x)\to f(x)$, and the author [same Doklady (N.S.) 60, 333–336 (1948); these Rev. 9, 584] did the same for a wider class of interpolation points. However, in these processes m_n/n did not tend to 1, and the principal aim of the present paper is to show that this is the general situation. Let

$l_k^{(n)}(x)$ be the fundamental Lagrange polynomials for the given $(x_k^{(n)})$, and seek polynomials of the form

$$(*)\quad A_n(f,x)=L_n(f,x)+\sum_{i=1}^{\sigma}[A_i^{(n)}(f)-f(x_{k_i}^{(n)})]l_{k_i}^{(n)}(x),$$

where $A_i^{(n)}(f)$ are linear functionals. The author's theorem 2 [corollary of a theorem 1 referring to a more general $A_n(f,x)$] states that if there is an L-integrable $g(x)\geqq 0$ for which

$$\int_{-1}^{1} l_k^{(n)}(x)\cos(j\cos^{-1}x)g(x)dx=O(1/n)$$

and if $\sigma_n=o(n)$ then it is impossible to find $A_i^{(n)}(f)$ such that $A_n(f,x)$ converges uniformly for every continuous $f(x)$. Hence the roots of a wide class of orthogonal polynomials, in particular those having weight function $g(x)$ with $g(x)(1-x^2)^{\frac{1}{2}}$ bounded, are inadmissible for an always convergent interpolation process $(*)$ with $\sigma_n=o(n)$. In the opposite direction the author states two theorems, involving restrictions on the $l_k^{(n)}(x)$ and on the modulus of continuity of $f(x)$, in which $A_n(f,x)\to f(x)$ and $\sigma=o(n)$.

R. P. Boas, Jr. (Evanston, Ill.).

Levi, Eugenio. Sopra un'applicazione dei polinomi di Bernstein all'approssimazione in media delle funzioni sommabili. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **9**, 242–246 (1950).

If $f(x)$ is integrable on $(0,1)$, and $F(x)$ is its indefinite integral, then the derivatives of the Bernstein polynomials for $F(x)$ converge in mean (of order 1) to $f(x)$. The author proves this first for step functions $f(x)$ and completes the proof by approximation of the general $f(x)$ by step functions.

R. P. Boas, Jr. (Evanston, Ill.).

Special Functions

***Lense, Josef. Kugelfunktionen.** Akademische Verlagsgesellschaft, Leipzig, 1950. xiii+294 pp. 26.00 DM.

The purpose of this book is to give a detailed presentation of the theory and of some of the more important applications of Legendre functions. The material goes far beyond what is usually found in text-books, although the book is not intended to be encyclopedic in character. Like the author's well-known "Reihenentwicklungen in der mathematischen Physik" [2d ed., de Gruyter, Berlin, 1947; these Rev. 11, 105], the book under review has the distinction of a sound and rigorous presentation on a comparatively elementary level, gives detailed proofs, and it is altogether very readable. The whole book is organised in three parts. Part I (96 pp.) deals with Legendre functions (and associated Legendre functions) with integer degree and order, and assumes only a good course on advanced calculus as a prerequisite. Part II (117 pp.) introduces Legendre functions whose parameters are arbitrary complex numbers. In this part it is natural to expect from the reader more knowledge of complex variable theory. Part III (78 pp.) contains selected applications.

In Part I, § 1 introduces preliminary matter on Laplace's equation in three dimensions, and some first information on solid spherical harmonics and spherical surface harmonics. Accordingly, in § 2, Legendre polynomials are defined by the relation

$$P_n\left(\frac{z}{r}\right)=\frac{(-)^n}{n!}r^{n+1}\frac{\partial^n r^{-1}}{\partial z^n},\qquad r^2=x^2+y^2+z^2.$$

The generating function, explicit representations (including integral representations), Rodrigues' formula, and recurrence relations are all obtained from this definition. In § 3 associated Legendre functions are introduced, defined in terms of derivatives of Legendre polynomials. P_n^m denotes Ferrer's associated Legendre function of the first kind (which is often denoted by T_n^m in English books). Hobson's associated Legendre function of the first kind is introduced later (in § 5) and is denoted by $\mathfrak{P}_n^m$. The very useful convention used in Jahnke-Emde, Magnus-Oberhettinger, and other important German publications of using Gothic letters for the definitions of Legendre functions appropriate for the cut plane, and the corresponding Latin letters for the definitions appropriate on the cut [that is on the interval $(-1,1)$] is adhered to in the present book. § 4 takes up the expansion of an arbitrary function defined on $(-1,1)$, in a series of Legendre polynomials. It is characteristic of the spirit of the whole book that the expansion theorem is proved only for a function $f(x)$ which is sectionally continuous and sectionally monotonic in the interval $(-1,1)$ and such that $(1-x^2)^{-\frac{1}{4}}|f(x)|$ is integrable over that interval; but under these rather restrictive assumptions the proof is carried through in a sound and convincing manner, and in great detail. § 5 introduces Legendre functions of the second kind and investigates their properties. Continued fractions, mechanical quadrature, and the occurrence of Legendre functions as characteristic functions of Legendre's differential equations are also discussed in this section. Spherical surface harmonics, their Gauss-Maxwell generation by differentiation, and the addition theorem of Legendre polynomials are in § 6; and § 7 with the expansion of arbitrary functions in a Laplace series (of surface harmonics) together with the solution of the boundary value problems of potential theory for the sphere closes part I.

In Part II, § 1 is devoted to the gamma function, and § 2 to the hypergeometric series. In § 3 the differential equation of associated Legendre functions, with arbitrary values of the parameters, is integrated by contour integrals. This leads to the definition of $\mathfrak{P}_\nu^\mu$ and $\mathfrak{Q}_\nu^\mu$ as contour integrals, to the definition of P_ν^μ and Q_ν^μ in terms of $\mathfrak{P}_\nu^\mu$ and $\mathfrak{Q}_\nu^\mu$, and to numerous integral representations and series expansions. This is the longest section in the whole book, and the multitude of series and integral representations will probably bewilder the beginner: yet experts are likely to agree that all the results presented in this section are really necessary, and that they are arranged as well and set forth as lucidly as the nature of the subject permits. § 4, on asymptotic expansions, is the second longest, and one of the most advanced, sections of the whole book, and it contains a most thorough and detailed investigation of Legendre functions by the method of steepest descents in addition to the better known asymptotic results. In view of some important applications it deserves special praise that the asymptotic behaviour of Legendre functions is discussed not only for real, but also for complex ν. § 5 gives the addition theorems for Legendre functions of the first and second kind for general degree and zero order.

Part III. § 1 brings applications to various problems relating to spheroids. § 2 is concerned with applications to geomagnetism, and the fitting of spherical surface harmonics to experimental data. § 3 introduces prolate and oblate spheroidal, and toroidal coordinates, and expresses solutions of Laplace's equation in these coordinates in terms of Legendre functions. It also contains a brief mention of conal harmonics. In § 4 Sommerfeld's work on the propagation

of electromagnetic waves around the earth is discussed, and the discussion includes Watson's transformation of the series into a contour integral together with the evaluation of this integral by van der Pol and Bremmer [Philos. Mag. (7) **24**, 141–176, 825–864 (1937)]. Lastly, § 5 discusses Laplace's equation in multi-dimensional spaces, introduces Gegenbauer polynomials and discusses some of their properties.

There is a name and subject index. References to literature are given throughout the text in footnotes, but some readers will regret that there is no list at the end of the most important books on the theory and application of Legendre functions. MacRobert's "Spherical Harmonics" [2d ed., Methuen, London, 1947; these Rev. **9**, 183] is not mentioned at all. Some of the topics that are not mentioned in the book are: the expansion of an analytic function of a complex variable in a series of Legendre functions, general Legendre functions when the sum or difference of degree and order is an integer, Green's functions for partial differential equations expressed in terms of Legendre functions, and the integral representation of an arbitrary function (of a real variable) in terms of conal harmonics. Even though a full-scale discussion of these topics may have been inopportune, one would have expected a brief mention of at least the last one of them. All in all an excellent and well-written book. In view of its comparatively moderate size it is likely to replace in some measure Hobson's "The Theory of Spherical and Ellipsoidal Harmonics" [Cambridge Univ. Press, 1931] as the standard work of reference.

A. Erdélyi (Pasadena, Calif.).

Beckenbach, E. F., Seidel, W., and Szász, Otto. Recurrent determinants of Legendre and of ultraspherical polynomials. Duke Math. J. **18**, 1–10 (1951).

This paper generalizes in several directions Turán's inequality that $\Delta_n(x)\leqq 0$ for $n\geqq 1$, $-1\leqq x\leqq 1$ (notation below). For $\lambda>-\frac{1}{2}$ let $P_\nu^{(\lambda)}(x)$ denote the ultraspherical polynomials, and let $p_\nu^{(\lambda)}(x)=P_\nu^{(\lambda)}(x)/P_\nu^{(\lambda)}(1)$. The authors prove that for $\lambda>0$ each n-rowed minor of the infinite determinant $|p_{i+j}^{\lambda}(x)|$ $(i, j=0, 1, 2, \cdots)$ is positive for $x>1$, has a zero of order $\frac{1}{2}n(n-1)$ at $x=1$, and is either even or odd as a function of x. The principal minor $i, j=0, 1, \cdots, n-1$ is evaluated explicitly for $\lambda>0$, as are certain other minors for $\lambda=\frac{1}{2}$ (Legendre polynomials). For $\lambda>-\frac{1}{2}$, $n\geqq 1$, $-1\leqq x\leqq 1$ it is proved that

$$(2n+1)[p_n^{(\lambda)}(x)]^2\geqq(2n-1)p_{n-2}^{(\lambda)}(x)p_{n+2}^{(\lambda)}(x)$$

and that $(2n+3)[p_{n+1}^{(\lambda)}(x)]^2\geqq(2n+1)p_{n-1}^{(\lambda)}(x)p_{n+3}^{(\lambda)}(x)$. For Legendre polynomials $P_n(x)$ let

$$\Delta_n(x)=P_{n-1}(x)P_{n+1}(x)-[P_n(x)]^2.$$

Noting that $\frac{1}{2}n(n+1)(d^2/dx^2)\Delta_n(x)=[(d/dx)P_n(x)]^2$ for $n\geqq 1$, $-\infty<x<\infty$, the authors prove that $\Delta_n(x)$ is a strictly convex, even function of x, with $\Delta_n(\pm 1)=0$, and they give some inequalities for $\Delta_n(x)$. They show that, for $x>1$, $\log P_\nu(x)$ is convex as a function of the continuous real variable ν.

G. E. Forsythe (Los Angeles, Calif.).

Gross, W. Sviluppo asintotico di alcuni integrali operanti su funzioni di Bessel. Giorn. Mat. Battaglini (4) **4(80)**, 39–49 (1951).

The author obtains an asymptotic representation of $\int_0^\infty(z\pm x)^{-1}J_m(x)J_n(z)dz$, $\Re(m+n)>1$, for large x from a differential equation (of the fourth order) satisfied by the integral.

A. Erdélyi (Pasadena, Calif.).

Barnett, M. P., and Coulson, C. A. The evaluation of integrals occurring in the theory of molecular structure. Parts I & II. Philos. Trans. Roy. Soc. London. Ser. A. **243**, 221–249 (1951).

The authors study various techniques for evaluating integrals which occur in molecular theory. The integrands of such integrals usually depend on distances from two or more fixed points, and accordingly they are called two-centre or multicentre integrals. A typical two-centre integral is

$$(1)\quad J=\int \exp(-\alpha r_a-\beta r_b)r_a^u r_b^v Y_p(\theta_a, \phi)Y_q(\theta_b, \phi)dV,$$

where a and b are two fixed points, (r_a, θ_a, ϕ) and (r_b, θ_b, ϕ) are spherical polar coordinates referred respectively to a and b, with the line joining a and b as the polar axis, α, β are positive parameters, u, v, p, q are integers, Y_p and Y_q are surface harmonics of the stated degrees, and integration is to be extended over the whole space.

First it is shown how to reduce J to a combination of integrals of the form

$$(2)\quad J(k, l, m)=\int \exp(-\alpha r_a-\beta r_b)\cdot(\cos\theta_a)^k r_a^{l-1}r_b^{m-1}dV.$$

For the evaluation of these integrals the authors introduce ζ-functions defined by the expansion

$$(3)\quad r_b^{m-1}e^{-\beta r_b}=(r_a\rho)^{-1}\sum_{n=0}^{\infty}(2n+1)P_n(\cos\theta_a)\zeta_{m,n}(\beta, r_a; \rho)$$

where ρ is the distance ab. The $\zeta_{0,n}$ are products of Bessel functions, and the other ζ functions can be obtained by successive differentiations with respect to β. With the expansion (3) the evaluation of (2) can be reduced to standard integrals and to certain Z-functions defined by

$$(4)\quad Z_{m,n,l+\frac{1}{2}}(\kappa, \tau)=\int_0^\infty e^{-\kappa t}\zeta_{m,n}(1, t; \tau)t^{l+\frac{1}{2}}dt.$$

Both numerical integration and analytical reduction of the Z-functions are discussed. Auxiliary numerical tables (to 7S) of $I_{n+\frac{1}{2}}(x)$ and $K_{n+\frac{1}{2}}(x)$ are provided, and the application of the results to overlap, resonance, Coulomb, hybrid, and other two-centre integrals is given. Formulas are listed for more than 180 distinct integrals.

A. Erdélyi.

Harmonic Functions, Potential Theory

Frostman, Otto. Sur les distributions vectorielles de masses. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **20**, 192–198 (1950).

Soit, dans l'espace euclidien R^3, une mesure μ à valeurs vectorielles, c.-à.-d. définie par ses 3 composantes scalaires μ_1, μ_2, μ_3. Son potentiel newtonien est une fonction vectorielle $\mathbf{A}(x)=(1/4\pi)\int_{R^3}|x-y|^{-1}d\mu(y)$. L'auteur démontre la formule

$$\int_{R^3}(|\operatorname{rot}\mathbf{A}(x)|^2+|\operatorname{div}\mathbf{A}(x)|^2)dx = (1/4\pi)\int\int|y-z|^{-1}d\mu(y)d\mu(z).$$

Cette formule, qui suppose que les intégrales considérées existent, s'étend au cas où l'on suppose que μ est remplacée par une distribution $\mathbf{T}$ de composantes T_i d'énergie new-

tonienne finie. D'ailleurs l'intégrale du 2e membre est la somme des énergies $\|T_i\|^2$, et la formule se déduit alors de celle de Deny [Acta Math. 82, 107–183 (1950), p. 154, formule 6, formule classique, traitée dans cet ouvrage pour les distributions d'énergie finie; ces Rev. 12, 98]. On en déduit la possibilité d'introduire un espace de Hilbert des distributions d'énergie finie vectorielles $\mathbf{T}$, avec cette norme $\|\mathbf{T}\|^2=\sum_i\|T_i\|^2$. Divers exemples et applications sont donnés. *L. Schwartz* (Nancy).

Verblunsky, S. On a fundamental formula of potential theory. J. London Math. Soc. 26, 25–30 (1951).

Let C denote a simple closed, rectifiable, Jordan curve in the (x, y)-plane, and let $U(Q)$ be real, bounded, and harmonic inside C. Then for $P\varepsilon C$, it is known $\lim U(Q)$ exists almost everywhere on C (in the sense of linear measure on C) as $Q\to P$ in a sector of angle $<\pi$ whose line of symmetry is the inner normal n_P to C at P; the limit is denoted by $U(P)$. Then, according to Tsuji [Jap. J. Math. 18, 379–383 (1942); these Rev. 8, 203], if $g(P, Q)$ is the Green's function for the interior of C, $\partial g/\partial n_P$ exists almost everywhere on C and $U(Q)=(2\pi)^{-1}\int_C U(P)[\partial g/\partial n_P]ds_P$. Tsuji's proof is incomplete; the present author gives a complete proof of the result. *M. Reade* (Ann Arbor, Mich.).

***Szegö, G. Studies on symmetrization.** Tech. Rep. no. 5, Navy Contract N6-ori-106 Task Order 5, Stanford University, Calif., 1950. i+112 pp.

L'auteur rassemble avec des compléments diverses propriétés et applications des opérations de symétrisation déjà développées surtout par Faber, Pólya, et Szegö [voir par exemple, Pólya, C. R. Acad. Sci. Paris. 230, 25–27 (1950); ces Rev. 11, 435; et l'analyse ci-dessous]. Ainsi dans le plan on passe d'une fonction u dans un domaine D à la frontière duquel elle prend une valeur constante, à une fonction $v(r)$ $(r=OM)$ prenant pour r la valeur constante de u sur la ligne niveau délimitant une région d'aire πr^2. On étudie d'abord pour un domaine annulaire D et l'intérieur G de la courbe limitante externe l'expression

$$D(u)=\int\int_D(|\operatorname{grad} u|^2+p(r)u^2)d\sigma-\int\int_G p(r)d\sigma;$$

$r=OM$, O origine entourée par la courbe limitante interne; u est nulle sur le bord interne, égale à 1 sur le bord externe, $p(r)$ est décroissante pour $r>0$, $rp(r)$ sommable au voisinage de $r=0$. La symétrisation minore $D(u)$ d'où un théorème de minimum généralisant un théorème de Carleman [Math. Z. 1, 208–212 (1918)] et s'appliquant à la notion de capacité. Développements analogues relatifs à la première autovaleur d'une membrane (minimum attient pour un cercle). Puis viennent des inégalités sur la capacité convenablement définie d'une couronne plane, sphérique ou cylindrique, des inégalités sur les "rayons" intérieur ou extérieur d'un compact plan, et sur les fréquences fondamentales des plaques vibrantes sous diverses hypothèses. Enfin sont étudiés divers types de symétrisation dans le plan et l'espace avec des applications à l'intégrale de Dirichlet $\int\int|\operatorname{grad} u|^2d\sigma$ et à la capacité (par exemple, la symétrisation plane de Steiner d'un domaine D relativement à une droite Δ consiste à faire glisser sur chaque normale à Δ les segments de D qu'elle intercepte pour les réunir en un segment centré sur Δ et cela conserve l'aire, minore le périmètre et le "rayon" extérieur (diamètre transfini), majore le "rayon" intérieur).

Un appendice rédigé en collaboration avec Schiffman étudie surtout dans le plan et pour des courbes convexes une transformation plus générale que la symétrisation de Steiner. *M. Brelot* (Grenoble).

Szegö, G. Über eine Verallgemeinerung des Dirichletschen Integrals. Math. Z. 52, 676–685 (1950).

The following two theorems are proved by means of symmetrization. I. Let $p(r)$ be a monotonically decreasing function defined for $r>0$ and such that $rp(r)$ is integrable at $r=0$. Let K_0 be a simple closed curve in the plane containing the origin, and K_1 another such curve containing K_0 in its exterior. Let B denote the domain between K_0 and K_1, G the domain interior to K_1. For a function u defined in B and assuming the values 1 and 0 on K_1 and K_0, respectively, set

$$D(u)=\int\int_B\{|\operatorname{grad} u|^2+p(r)u^2\}dxdy-\int\int_G p(r)dxdy \qquad (r^2=x^2+y^2).$$

The minimum of $D(u)$, for fixed areas of B and G, is smallest when K_0 and K_1 are concentric circles with center at the origin. II. Let $p(r)$ and K_0 have the same meaning as before, and let D denote the domain interior to K_0. Also, let $\lambda>0$ be the smallest eigenvalue of the equation $\Delta u+\lambda^2pu=0$ for the domain D and the boundary condition $u=0$ on K_0. For fixed area of D, λ is smallest when K_0 is a circle with center at the origin. For $p(r)=0$ the first theorem reduces to a result of Carleman [Math. Z. 1, 208–212 (1918)], the second to one of Faber [S.-B. Math.-Phys. Kl. Bayer. Akad. Wiss. 1923, 169–172]. *L. Bers* (Los Angeles, Calif.).

Górski, Jerzy. Remarque sur le diamètre transfini des ensembles plans. Ann. Soc. Polon. Math. 23, 90–94 (1950).

Leitner, Roman. Sur une propriété des ensembles plans de diamètre transfini nul. Ann. Soc. Polon. Math. 23, 183–189 (1950).

Terasaka, Hidetaka. Solution of a problem of M. F. Leja. Ann. Soc. Polon. Math. 23, 201–204 (1950).

Leja, F. Remarque sur la note précédente. Ann. Soc. Polon. Math. 23, 204–205 (1950).

Let $\rho(p, q)$ be the distance function, $g(p, q)=\log\rho(p, q)$ and $\frac{1}{2}n(n-1)g(p_1, \cdots, p_n)=\sum\sum_{i<j}g(p_i, p_j)$. For an infinite closed bounded plane set E let g_n be the maximum of $g(p_1, \cdots, p_n)$ for points $p_i\varepsilon E$ and let $q^{(n)}=(q_1^{(n)}, \cdots, q_n^{(n)})$ be an extremal set. Let $(n-1)g_i(q^{(n)})=\sum_{j\neq i}g(q_i^{(n)}, q_j^{(n)})$ and suppose the $q_i^{(n)}$ so ordered that $g_i(q^{(n)})\geqq g_j(q^{(n)})$ for $i<j$. Also let $ng_n(z)=\sum_i g(q_i^{(n)}, z)$ for z outside E.

Leja had shown [same Ann. 18, 4–11 (1945); these Rev. 8, 255] that (i) $g_1(q^{(n)})\to g(\tau)$ as $n\to\infty$, where τ is the transfinite diameter of E, (ii) if $\tau>0$ then $g_n(z)$ converges as $n\to\infty$, the limit being then a Green's function.

Gorski proves that $g_n(q^{(n)})\to g(\tau)$ if E is a sum of continua, but not if E is a segment plus a suitable isolated point.

Leitner constructs a denumerable set E, having only two limit points, for which $g_n(z)$ does not converge as $n\to\infty$.

Terasaka constructs a similar example to Leitner's for the 3-dimensional potential $g(p, q)=-1/\rho(p, q)$. Leja remarks that the analogue for this potential of his (ii) has not been proved. *H. D. Ursell* (Leeds).

Leja, F. Une méthode élémentaire de résolution du problème de Dirichlet dans le plan. Ann. Soc. Polon. Math. 23, 230–245 (1950).

Let D_m be a domain of the plane, containing the point at infinity as an interior point, and $\varphi(z)$ a real function con-

tinuous on F, the boundary of D_∞; it is assumed that $D_\infty+F$ has a proper complement, Δ. Let $Z^{(n)}=\{Z_0, Z_1, \cdots, Z_n\}$ be an arbitrary set of $n+1$ distinct points of F, and set

$$V(Z^{(n)})=\prod_{0\le j<k\le n}|Z_j-Z_k|,$$

$$L^{(j)}(z, Z^{(n)})=\prod_{k=0,\,k\neq j}^{n}\frac{z-Z_k}{Z_j-Z_k}, \qquad j=0, 1, \cdots, n,$$

$$V_\lambda(Z^{(n)})=V(Z^{(n)})\exp\left(-n\lambda\sum_{j=0}^{n}\varphi(Z_j)\right), \qquad \lambda>0.$$

Let $x^{(n)}=\{x_0^{(n)}, x_1^{(n)}, \cdots, x_n^{(n)}\}$ be a system of $n+1$ points of F (depending on λ) on which $V_\lambda(Z^{(n)})$ attains its maximum value, and set

$$\Phi^{(j)}(z, \lambda, x^{(n)})=L^{(j)}(z, x^{(n)})\exp\left(n\lambda\varphi(x_j^{(n)})\right), \quad j=0, 1, \cdots, n,$$

$$F_n(z, \lambda)=\sum_{j=0}^{n}|\Phi^{(j)}(z, \lambda, x^{(n)})|, \quad n=1, 2, \cdots.$$

The principal result given is that

$$\lim_{\lambda\to 0}\left\{\lim_{n\to\infty}\frac{1}{\lambda}\log\left[F_n(z, \lambda)\right]^{1/n}\right\}$$

exists and constitutes the solution of the Dirichlet problem for the bounded open set Δ and the boundary values $\varphi(z)$.

F. W. Perkins (Hanover, N. H.).

Maple, Clair G. The Dirichlet problem: Bounds at a point for the solution and its derivatives. Quart. Appl. Math. **8**, 213–228 (1950).

L'autore applica il metodo dell'ipercircolo di Synge [Proc. Roy. Soc. London. Ser. A. **191**, 447–467 (1947); questi Rev. **10**, 81] per ottenere delle limitazioni, in un punto interno al campo di integrazione, della soluzione del problema di Dirichlet per l'equazione $\Delta_2 u=0$, e delle derivate. Tali limitazioni sono ottenute anche nei punti appartenenti a parti piane delle frontiera e sono fatte applicazioni numeriche.

L. Amerio (Milano).

Inoue, Masao. Sur la méthode des médiations réitérées dans le problème de Dirichlet. Mem. Fac. Sci. Kyūsyū Univ. A. **5**, 41–53 (1950).

Let Ω denote a bounded domain, with frontier Γ, in 3-space, and let $f(Q)$ be real, continuous on Γ. Lebesgue [C. R. Acad. Sci. Paris, **154**, 335–337 (1912)] and Perkins [ibid. **184**, 182–183 (1927)] have shown that a certain sequence of iterated averages converges to the solution of the generalized Dirichlet problem for Ω, f. The present author extends both the method and results to solutions of the Poisson equation. *M. Reade* (Ann Arbor, Mich.).

Inoue, Masao. Sur les fonctions de noeud et leurs applications à l'intégration numérique des équations aux dérivées partielles. Mem. Fac. Sci. Kyūsyū Univ. A. **4**, 107–178 (1949).

The author replaces the equation $\Delta U+CU+\phi=0$, by the corresponding difference equation, $\Delta_h u+cu+\varphi=0$. Here c, $c\leqq 0$, and φ are given lattice functions. In continuation of some previous investigations [Proc. Imp. Acad. Tokyo **19**, 431–437 (1943); these Rev. **7**, 301] the author considers the lattice analogues of Green's function and of harmonic measure. He proves various formulas referring to the solution of boundary value problems for generalized Laplace and Poisson's equations and gives bounds for the difference between the solution of the difference and exact equations, corresponding to the same boundary value problem. A number of examples of the application of the difference method for solution of boundary value problem, mostly for rectangles, are given. Different procedures are used and the results obtained are compared. *S. Bergman.*

Karabegov, V.-K. I. On the stability of the Dirichlet problem for the equation $\Delta u+\lambda u=0$. Doklady Akad. Nauk SSSR (N.S.) **75**, 491–494 (1950). (Russian)

The author gives an example of a domain G_0 for which the Dirichlet problem $\Delta u+\lambda_0 u=0$ always has an unique solution in the classical sense; but for an arbitrary sequence of domains which approximate G_0 from the outside and for the same boundary function f there is given two essentially distinct sequences $\{U_{n,f}^{(1)}\}$ and $\{U_{n,f}^{(2)}\}$, depending on the method of continuation of f, which converge uniformly to the two different solutions $U_f^{(1)}$ and $U_f^{(2)}$ inside G_0.

C. G. Maple (Washington, D. C.).

Wintner, Aurel. On the Hölder restrictions in the theory of partial differential equations. Amer. J. Math. **72**, 731–738 (1950).

The author calls a function $u(x, y)$, defined and continuous in an open set D, a solution of the equation (1) $\Delta u+f(x, y)u=0$ if and only if $u_{x,x}$ and $u_{y,y}$ exist and satisfy (1). He proves that there exist functions $f(x, y)$ continuous on the entire (x, y)-plane such that there does not exist any domain D, on which (1) possesses some solutions $u(x, y)$ distinct from $u=0$. *C. Miranda* (Naples).

Wintner, Aurel. On a geometrical method of deriving three-dimensional harmonic flows from two-dimensional ones. Quart. Appl. Math. **9**, 102–105 (1951).

The problem considered is to generate 3-space harmonic functions $\psi(x, y, z)$ from given 2-space harmonic functions $\phi(u, v)$. The problem is solved by $\psi(x, y, z)=\phi(u, v)$ if (x, y, z) is on the straight line $\gamma(u, v)$, where the $\gamma(u, v)$ form an isotropic line congruence in the 3-space.

L. M. Milne-Thomson (Greenwich).

Reitan, Daniel Kinseth, and Higgins, Thomas James. Calculation of the electrical capacitance of a cube. J. Appl. Phys. **22**, 223–226 (1951).

The determination of the exact electrostatic capacitance of a cube is a famous unsolved problem. Pólya [Amer. Math. Monthly **54**, 201–206 (1947); these Rev. **8**, 514] has obtained bounds for this capacitance; it lies between .62211 and .71055 times the side a of the cube. The present authors obtain an estimate of this capacitance by approximating the conductor distribution, which has constant potential over the cube, in the following manner: a 6 by 6 checkerboard is laid off on each face of the cube. Each square A_i (216 in all) is provided with an unknown mass m_i distributed uniformly over it. The potential due to the mass on the square A_i is proportional to m_i. The total potential is computed at the center of each square and equated to a constant V_0. There results 216 linear equations in the m_i, and these because of the symmetry of the cube, reduce to 6. The m_i are determined and the estimate of the capacity is given by $\sum m_i/V_0=.6555a$ e.s.u. This result does indeed lie within the bounds established by Polya and Szegö, but there is of course no estimate of error and thus no guarantee of accuracy. *J. W. Green* (Los Angeles, Calif.).

Differential Equations

Ważewski, Tadeusz. Systèmes des équations et des inégalités différentielles ordinaires aux deuxièmes membres monotones et leurs applications. Ann. Soc. Polon. Math. 23, 112–166 (1950).

A set of functions $f_i(t, y^1, \cdots, y^n)$, $i=1, \cdots, n$, has property H if $a_\nu \leqq b_\nu$ for $\nu \neq i$ implies $f_i(t, a_1, \cdots, a_n) \leqq f_i(t, b_1, \cdots, b_n)$. It has property K if each f_i is a nondecreasing function of each y^ν with $\nu \neq i$. Kamke [Acta Math. 58, 57–85 (1932), theorem 7] has shown that if the f_i have property K in an open set Ω then the system of differential equations (A) $dy^i/dt = f_i(t, y^1, \cdots, y^n)$ has, through each $(t_0, a_1, \cdots, a_n) \in \Omega$, an integral which is maximal on some interval $t_0 \leqq t < \beta$. A counterexample is given for Kamke's statement that β can be chosen so that the maximal integral approaches the boundary of Ω as t approaches β.

Among the positive results are the following. (1) If property K is replaced by property H then Kamke's result is correct in toto. (2) If (i) the f_i have property H in Ω, (ii) $(\tau^1(t), \cdots, \tau^n(t))$ is a maximal integral of (A) on the interval $t_0 \leqq t < \alpha$, (iii) the lower right derivate $\underline{D}_+\psi^i(t) \leqq f_i(t, \psi^1(t), \cdots, \psi^n(t))$ for $t_0 \leqq t < \alpha$, (iv) $(t, \tau^1(t), \cdots, \tau^n(t)) \in \Omega$ and $(t, \psi^1(t), \cdots, \psi^n(t)) \in \Omega$ for $t_0 \leqq t < \alpha$, and (v) $\psi^i(t_0) \leqq \tau^i(t_0)$ for $i=1, \cdots, n$, then $\psi^i(t) \leqq \tau^i(t)$ for $i=1, \cdots, n$ and $t_0 \leqq t < \alpha$. (3) The conclusion, the f_i have property H in Ω follows from the hypothesis: whenever $(t_0, b_1, \cdots, b_n) \in \Omega$ and a solution $(\psi^1, \cdots, \psi^n)$ of (A) are given so that $\psi^i(t_0) \leqq b_i$ then there is an $\epsilon > 0$ and a solution $(\tau^1, \cdots, \tau^n)$ of (A) such that $\tau^i(t_0) = b_i$ and $\psi^i(t) \leqq \tau^i(t)$ for $t_0 \leqq t < t_0 + \epsilon$. (4) If the hypothesis of (3) holds with $t_0 \leqq t \leqq t_0 + \epsilon$ replaced by $t_0 - \epsilon \leqq t \leqq t_0 + \epsilon$ then f_i depends on t and y^i alone.

Among the applications is a method for estimating the error in using solutions of one system of differential equations as approximations to those of another.

F. M. Stewart (Providence, R. I.).

Szarski, J. Sur les systèmes majorants d'équations différentielles ordinaires. Ann. Soc. Polon. Math. 23, 206–223 (1950).

Analogues of a theorem of Ważewski [see (3) in the preceding review] are proved. E.g. Let Ω be an open set in an $(n+1)$-dimensional Euclidean space such that for each t_0 the set of $(y_1, \cdots, y_n)$ for which $(t_0, y_1, \cdots, y_n) \in \Omega$ is convex. Let $f_i(t, y_1, \cdots, y_n)$, $i=1, \cdots, n$, be continuous in Ω together with their first partial derivatives. Then in order that through each $(t_0, a_1, \cdots, a_n) \in \Omega$ there is a solution $(\tau_1, \cdots, \tau_n)$ of $dy_i/dt = f_i(t, y_1, \cdots, y_n)$ such that if (i) g_i is continuous on Ω, (ii) $g_i(t, y_1, \cdots, y_n) \leqq f_i(t, y_1, \cdots, y_n)$ on Ω, (iii) $\psi_i(t_0) = a_i$, and (iv) $d\psi_i/dt = g_i(t, \psi_1(t), \cdots, \psi_n(t))$ for $t_0 \leqq t < t_0 + \alpha$, $\alpha > 0$, then there is an $\epsilon > 0$ such that $\psi_i(t) \leqq \tau_i(t)$ for $t_0 \leqq t < t + \epsilon$, it is both necessary and sufficient that $\partial f_i/\partial y_j \geqq 0$ for $j \neq i$ and $(t, y_1, \cdots, y_n) \in \Omega$.

F. M. Stewart (Providence, R. I.).

Zwirner, Giuseppe. Un criterio di confronto per equazioni differenziali del primo ordine. Ann. Univ. Ferrara. Parte I. 7, 213–216 (1948).

The author gives an extension of a theorem due to Scorza Dragoni [Rend. Sem. Mat. Univ. Padova 12, 30–50 (1941); these Rev. 8, 207] concerning the comparison of solutions of two ordinary differential equations of the first order.

L. A. MacColl (New York, N. Y.).

Mikołajska, Z. Sur les transformations des systèmes d'équations différentielles linéaires. Ann. Soc. Polon. Math. 23, 272–278 (1950).

Among a large class of changes of dependent variables only $x_i = \sum_{k=1}^{n} e^{\tau\gamma_{ik}}\xi_k$, $t = \tau$ transforms systems of differential equations of the form $x_i' = \sum_{j=1}^{n} a_{ij}x_j$ into ones of similar form.

F. M. Stewart (Providence, R. I.).

Hartman, Philip, and Wintner, Aurel. On the classical transcendents of mathematical physics. Amer. J. Math. 73, 381–389 (1951).

In a previous paper [same J. 71, 367–372 (1949); these Rev. 10, 711] the authors have given sufficient conditions in order that a linear second order differential equation have a solution which is completely monotone in an interval (a, b). In the present paper these criteria are applied to the associated Legendre equation (toroidal functions), Gauss's equation (hypergeometric functions) and Kummer's equation (confluent hypergeometric functions) which possess completely monotone solutions when the parameters are restricted to specified intervals. Questions of uniqueness are decided. The authors state that the monotony properties cannot be read off from available integral representations of the solutions. The existence of various representations of the form $f(z)\int_0^\infty [g(z)]^t d\phi(t)$ is proved under specified conditions.

E. Hille (New Haven, Conn.).

Ascoli, Guido. Osservazioni sopra alcune questioni di stabilità. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 9, 129–134 (1950).

The author discusses some questions connected with the boundedness of solutions of the vector-matrix equation $dy/dt = (A + B(t))y$ where $B(t)$ is small in some sense as $t \to \infty$, and their relation to known results.

R. Bellman.

Massera, José L. The existence of periodic solutions of systems of differential equations. Duke Math. J. 17, 457–475 (1950).

Let differential equations (1) $dx_i/dt = X_i(x_1, \cdots, x_m, t)$, $i=1, \cdots, m$, be given, where the X_i are smooth for all $x=(x_1, \cdots, x_m)$ and t and have period 1 in t. The author establishes several results, partly negative, concerning the existence of solutions $x=x(t)$ of (1) having period 1 or multiples thereof (subharmonics). Thus, for $m=2$, a solution of period 1 is assured if all solutions $x(t)$ can be continued for arbitrarily large t and if at least one solution is bounded for t sufficiently large. Again for $m=2$ it is shown that there exist systems (1) having a prescribed number $N(q)$ of solutions of period q, $q=1, 2, \cdots$, and no other periodic solutions; here $N(q)$ is 0 or ∞ or a multiple of q. A similar statement holds for $m>2$, even with analytic X_i and the requirement that all solutions $x(t)$ be defined for $-\infty < t < \infty$; in particular one can have $N(q)=0$ and still have at least one bounded solution.

W. Kaplan (Ann Arbor, Mich.).

Erugin, N. P. Some general questions of the theory of stability of motion. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 227–236 (1951). (Russian)

This paper contains a number of rather unrelated results concerning the vector equation (*) $\dot{x} = X(x, t)$. In § 1 there is given a simple criterion ensuring that each solution of (*) exists for all t. In § 2, x is 2-dimensional and X is independent of t; two theorems concerning the existence of limit cycles are proved, the second of which is the following. Let the origin be the sole point of equilibrium, and assume that it is asymptotically stable; let L be a ray from the origin

which is cut by all trajectories in the same direction; let there be a trajectory not tending to the origin and cutting L at least twice; then a limit cycle exists. X and x are general again in § 3; the basic theorem here states that if A is an open set containing all trajectories which start in it then the boundary of A has the same property. (A uniqueness hypothesis would seem to be necessary for the argument.) A number of corollaries are deduced from this result in §§ 3–5, including theorems concerning the boundary of the region of asymptotic stability, and applications to limit cycles. *J. G. Wendel* (Santa Monica, Calif.).

Lur'e, A. I. On strictly unstable regulating systems. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 251–254 (1951). (Russian)

The equations of motion of a servomechanism are put in the canonical form $z_i=\mu_i z_i+\varphi(\sigma)$, $\sigma=\sum \nu_i z_i$, $i=1, 2, \cdots, n+1$, and a Liapounoff function is constructed for the system. *J. G. Wendel* (Santa Monica, Calif.).

Lur'e, A. I. On the problem of the stability of regulating systems. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 67–74 (1951). (Russian)

The author considers the stability of the particular system

$$dx_i/dt=\lambda_i x_i+f(s),\quad ds/dt=\sum_{k=1}^{n} b_k x_k-hf(s),\quad i=1, 2, \cdots, n,$$

where $f(s)$ satisfies the condition $sf(s)>0$, and $\Re(\lambda_i)<0$. *R. Bellman* (Stanford University, Calif.).

Malkin, I. G. On the theory of stability of regulating systems. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 59–66 (1951). (Russian)

Using the second method of Liapounoff, the author discusses the stability of the system $dy/dt=Ay+f(u)h$, where A is a constant matrix, h is a constant vector and f is a scalar function of a linear function of the components of y. *R. Bellman* (Stanford University, Calif.).

Neigauz, M. G., and Lidskiĭ, V. B. On the boundedness of the solutions of linear systems of differential equations with periodic coefficients. Doklady Akad. Nauk SSSR (N.S.) **77**, 189–192 (1951). (Russian)

Let $H(t)$ be a $2k$-rowed symmetric square matrix of piecewise continuous functions periodic with period ω. Let $\|H(t)\|=\max_i|\lambda_i(t)|$, where the $\lambda_i(t)$ are the characteristic roots of $H(t)$. Let I be the matrix $\left(\begin{smallmatrix}0 & E\\ -E & 0\end{smallmatrix}\right)$, where E is the k-dimensional identity. The authors study systems of the form $y'=IH(t)y$; among other things they are concerned with the property of strong stability. Such systems are called strongly stable in case there is an $\epsilon>0$ such that for any matrix $Q(t)$ satisfying the conditions on H above, if $\|Q(t)\|<\epsilon$, then all solutions of $y'=I(H+Q)y$ are bounded. Criteria for strong stability are obtained. *J. G. Wendel* (Santa Monica, Calif.).

Farnell, A. B., Langenhop, C. E., and Levinson, N. Forced periodic solutions of a stable non-linear system of differential equations. J. Math. Physics **29**, 300–302 (1951).

Consider the system $\dot{x}=Ax+f(x, \omega t)+kb(\omega t)$, where x is an n-vector, A is a constant matrix whose characteristic roots have negative real parts, $b(t)$ is continuous, has period 1 and mean value 0, k and ω are constants, $f(x, t)$ is continuous, periodic of period 1 in t, $f(0, t)=0$, and for any $\eta>0$, $|f(x, t)-f(y, t)|\leqq\eta|x-y|$ whenever $|x|$, $|y|$ are small enough. The authors prove: If k is sufficiently small or ω is sufficiently large, the system has a periodic solution $p(t)$ of period $1/\omega$. Moreover there exists a constant $\rho>0$ such that $|p(t)|\leqq\rho k/(1+\omega)$. The solution $p(t)$ is stable and indeed there exists an $\epsilon>0$ such that any solution $x(t)$ which at $t=t_0$ satisfies $|x(t_0)|\leqq\epsilon$ also satisfies $x(t)-p(t)\to 0$ as $t\to\infty$ providing $k/(1+\omega)$ is small. *J. L. Massera.*

Haag, Jules. Cols, noeuds et foyers. Bull. Sci. Math. (2) **74**, 167–192 (1950).

The author considers the perturbed linear system of real differential equations $dx_j/dt=\sum_{k=1}^{m} a_{jk}x_j+f_j(x_1, \cdots, x_m; t)$, where $j=1, \cdots, m$, $f_j(0, \cdots, 0; t)\equiv 0$ and $f_j(x_1, \cdots, x_m; t)$ satisfies a uniform Lipschitz condition with respect to $x_1, \cdots, x_m$, with a Lipschitz constant which is independent of t and which is of the order $O(R^\alpha)$, $\alpha>0$, in the vicinity $|x_k|<R$ of the origin $x_1=\cdots=x_m=0$. He obtains an extension of Poincaré's theory of solutions near a singular point for the case of an arbitrary m (rather than $m=2$) and for nonanalytic perturbations f_j. [For the analytic case, cf. Chazy, same Bull. (2) **45**, 270–280 (1921); **56**, 79–104 (1932).] In addition f_j depends on t; but the heavy assumption, mentioned above, concerning the independence of t in the estimates on f_j, implies that the occurrence of t introduces no new complications. Assuming that the matrix (a_{jk}) is in a normal form and using the method of successive approximations, the author obtains a classification into families, defined in terms of the real parts of the characteristic numbers of (a_{jk}) and of initial conditions, of the solutions satisfying $x_j(t)\to 0$, as $t\to\infty$, for $j=1, \cdots, m$. Further asymptotic properties, as $t\to\infty$, of solutions in these families are obtained. For example, if the characteristic numbers of (a_{jk}) are all (real and) negative [node], then all solution curves reach the origin for $t=\infty$ with a tangent; the possible tangent lines of the different solutions depend on the elementary divisors of (a_{jk}). [The case $m=2$ has been studied in detail by Perron, Math. Z. **15**, 121–146 (1922); **16**, 273–295 (1923). Some of the simpler results of the author can be obtained more easily by the methods of Wintner, Amer. J. Math. **69**, 815–824 (1947); these Rev. **9**, 285.] *P. Hartman* (Baltimore, Md.).

Haag, Jules. Sur la synchronisation des systèmes oscillants non linéaires. Ann. Sci. École Norm. Sup. (3) **67**, 321–392 (1950).

This is an elaborate study of the following problem: A nonlinear dynamical system, such that its autonomous motions include just one periodic motion, is subjected to perturbing forces which either do not depend explicitly upon the time or are periodic with respect to the time; it is required to determine whether or not the perturbed motions of the system include a stable periodic motion, and to determine the period of such a motion if it exists. The general problem is treated in chapter I. In the remaining chapters the general results are applied to several more or less particular cases, including the general case of a system of one degree of freedom, the case of an Abelé oscillator, and the cases of various relaxation oscillators. The discussion throughout the paper depends heavily upon methods and results which the author has presented in earlier papers [same Ann. (3) **60**, 35–64, 65–111 (1943); **61**, 73–117 (1944); **64**, 285–338 (1948); **65**, 299–335 (1948); Bull. Sci. Math. (2) **73**, 123–134 (1949); these Rev. **7**, 299; **10**, 377; **11**, 111, 514], and an acquaintance with these papers is essential to an understanding of the details. It should be noted that Bohnenblust and Levinson, in their reviews [these Rev. **11**, 111, 514], have pointed out errors in two of the papers cited

above. It appears that the erroneous results have been used in this work without correction. *L. A. MacColl.*

Haag, Jules. À propos de l'équation de Mathieu. C. R. Acad. Sci. Paris **232**, 661–663 (1951).

The author gives criteria for the stability of the periodic solutions of differential equations of the type

$$\theta''+\theta=\lambda[H(\theta,\theta')\cos\omega t+F(\theta,\theta')]$$

where λ is a positive parameter. These criteria are easy consequences of the author's more general theory [Ann. Sci. École. Norm. Sup. (3) **64** (1947), 285–338 (1948); these Rev. **10**, 377; Ann. Françaises de Chronométrie (2) **1**, 201–264 (1947), p. 218]. Various special forms of $H(\theta,\theta')$ and $F(\theta,\theta')$ are examined. As a special case the results of Minorsky [same C. R. **231**, 1417–1419 (1950); these Rev. **12**, 413] are regained. *W. Wasow.*

McLachlan, N. W. Non-linear differential equation having a periodic coefficient. Math. Gaz. **35**, 32–36 (1951).

The vibration equation of a weighted string attached to an electromagnetically driven reed or tuning fork and subject to energy dissipation may be written in the form $y''+2ky'+\{(a+by^2)-2q\cos 2z\}y=0$, where z is proportional to the time, and y represents the lateral displacement. The equation is studied under various simplifying assumptions: first, that $b=k=0$; then, that $k=0$ and b is small; and finally, that k and b are small. Approximation methods are used, and the results of the latter two cases are found to be in good agreement with experiment.

J. G. Wendel (Santa Monica, Calif.).

Ascoli, Guido. Osservazioni sopra alcune questioni di stabilità. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **9**, 210–213 (1950).

The author shows by means of ingenious transformations that all the solutions of $u''+(1+kt^{-1}\sin at)u=0$ are bounded for $a\neq 2$, while this result is not true for $a=2$. A separate discussion is necessary in this case since it eludes the known stability criteria which require absolute integrability or bounded variation, due to Ascoli. *R. Bellman.*

Williams, John. Small oscillations with damping. Math. Gaz. **35**, 29–31 (1951).

The author feels that "familiarity with approximations is an essential part of the early training of a good applied mathematician." An illustration of the sort of problem to which students might profitably be exposed is presented. The equation $\ddot{x}+\frac{1}{2}\mu\dot{x}|\dot{x}|+n^2x=0$ represents the motion of a particle subject to a linear restoring force and a damping term proportional to the square of the velocity. Assuming the initial conditions $t=0$, $x=a>0$, $\dot{x}=0$, and μa small the position and time of the first stationary point are estimated, using two different well-known methods of approximation; the results are seen to agree through terms of order μ^2.

J. G. Wendel (Santa Monica, Calif.).

Cecconi, Jaurès. Su di una equazione differenziale di rilassamento. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **9**, 38–44 = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **284** (1950).

Using the Poincaré-Bendixon plane the author considers the differential equation $y''+y'|y'|-qy'+y-p^2y^3=0$, where p and q are positive constants. He proves there exists a $p_0>0$ such that if $p<p_0$ the equation has a periodic solution (aside from the trivial solution $y=0$). *N. Levinson.*

Schäfke, Friedrich Wilhelm. Über die Stabilitätskarte der Mathieuschen Differentialgleichung. Math. Nachr. **4**, 175–183 (1951).

For the Mathieu differential equation the author derives results on the asymptotic behavior of the characteristic exponents, certain product relations between the points of intersection of characteristic curves of the stability chart with pairs of parallel lines, and formulas useful for the calculation of the characteristic exponents. The given treatment is based on the Weierstrass product expansion of an entire function. *W. T. Reid* (Evanston, Ill.).

Verholomov, D. F. Equations of the form

$$y'''=R(y',y,x)y''^2$$

with invariant critical points. Ukrain. Mat. Žurnal **2**, no. 2, 84–93 (1950). (Russian)

Equations of the form $y'''=y''^2\{A/(y'+ay^2)+B/(y'+by^2)\}$ which may have the title property are classified. Since only necessary conditions are considered the final list may be too large. *J. G. Wendel* (Santa Monica, Calif.).

Funk, Paul. Über das Newtonsche Abbildungsgesetz in der Elektronenoptik. Acta Physica Austriaca **4**, 304–308 (1950).

Il problema matematico da risolvere è il seguente: Quando avviene che due zeri successivi t, $\bar{t}$ dell'equazione differenziale $d^2y/dt^2+p(t)y=0$ siano legati necessariamente dalla relazione: $at\bar{t}+bt+c\bar{t}+d=0$? Se $y_1(t)$ e $y_2(t)$ sono due soluzioni linearmente indipendenti e si interpretano come coordinate cartesiane ortogonali in un piano, al variare di t si ottiene una curva. Se si passa alle coordinate polari r e φ, la funzione $z(\varphi)=1/r$ obbedisce a un'equazione differenziale di Hill.

G. Toraldo di Francia (Firenze).

Moretti, Mario. Una formula e sua applicazione alla risoluzione di una classe di equazioni differenziali lineari. Rivista Mat. Univ. Parma **1**, 471–473 (1950).

Let $P(x)$ be a polynomial of degree n, and put

$$L[f(x)]=\sum_{r=0}^{n}(r!)^{-1}P^{(r)}(x)f^{(r)}(x).$$

The author observes that there is exactly one set of constants a_i with which

$$L[f(x)]=e^{-\frac{1}{2}x^2}\sum_{i=0}^{n}a_i\{e^{\frac{1}{2}x^2}f(x)\}^{(i)}$$

is an identity, and uses this remark to solve the differential equation $L[f(x)]=0$. Superscripts in parentheses indicate derivatives. *A. Erdélyi* (Pasadena, Calif.).

Imai, Isao. Asymptotic solutions of ordinary linear differential equations of the second order. Physical Rev. (2) **80**, 1112 (1950).

This is merely a brief summary of formulas from the author's earlier paper [Physical Rev. (2) **74**, 113 (1948); these Rev. **10**, 41]. *R. E. Langer* (Madison, Wis.).

Marčenko, V. A. On the transformation formulas generated by a linear differential operator of the second order. Doklady Akad. Nauk SSSR (N.S.) **74**, 657–660 (1950). (Russian)

In the notation of an earlier paper [same Doklady **74**, 185–188 (1950); these Rev. **12**, 502], the author considers the Weyl expansion theorem

$$E(\lambda)=\int_{-\infty}^{\infty}f(x)\tilde{\omega}_\lambda(\lambda,x)dx,\qquad f(x)=\int_{-\infty}^{\infty}E(\lambda)\tilde{\omega}_\lambda(\lambda,x)d\rho(\lambda)$$

where $f(x)\varepsilon L^2(0, \infty)$. Lemmas concerning the growth of $\rho(\lambda)$ are given extending earlier results of Levitan [ibid. 71, 605–608 (1950); these Rev. 11, 720]. Formulas relating $\rho(\lambda)$ and $c(\lambda)=\int_0^\infty f(x)\cos\lambda x\,dx$ are obtained as are other related results. *N. Levinson* (Cambridge, Mass.).

Case, K. M. Singular potentials. Physical Rev. (2) **80**, 797–806 (1950).

To consider the Schroedinger equation with a potential that becomes infinite at $x=0$, the author analyses the equation

$$u''+(\lambda x^{-n}-\eta^2)u=0, \qquad 0<x<\infty,$$

in which λ is a fixed constant and η^2 the parameter. When $n=2$ the solution of this which is bounded for large x, has for small x the form $x^{\frac{1}{2}}\cos(\lambda'\log x+B(\eta))$ with $\lambda'=(\lambda-\frac{1}{4})^{\frac{1}{2}}$. The primary concern of the paper is with the fact that to require of $B(\eta)$ that it have a value independent of η is to impose orthogonality upon the solutions. It is thus a basis for determining eigenvalues for the equation. It is shown that the considerations can be adapted to the case $n>2$, and generally to familiar cases of quantum mechanics in which the potential becomes infinite with sufficient rapidity as $x\to\infty$. *R. E. Langer* (Madison, Wis.).

Titchmarsh, E. C. A relation between Green's functions. J. London Math. Soc. **26**, 31–36 (1951).

If $G_1(x, \xi, \lambda)$ and $G_2(y, \eta, \lambda)$ are the Green's functions of two differential equations

$$d^2\varphi/dx^2+\{\lambda-q_1(x)\}\varphi=0, \quad d^2\varphi/dy^2+\{\lambda-q_2(y)\}\varphi=0,$$

under conditions for which the two spectra are bounded below, and if $G(x, y, \xi, \eta, \lambda)$ is the Green's function of the partial differential equation

$$\partial^2\varphi/\partial x^2+\partial^2\varphi/\partial y^2+\{\lambda-q_1(x)-q_2(y)\}\varphi=0,$$

it is known that formally

$$G(x, y, \xi, \eta, \lambda)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}G_1(x, \xi, \mu)G_2(y, \eta, \lambda-\mu)d\mu.$$

The paper proves that this relation holds when the right-hand member is uniformly convergent. *R. E. Langer.*

Pini, Bruno. Sui sistemi di equazioni lineari del primo ordine ai differenziali totali. Boll. Un. Mat. Ital. (3) **5**, 255–264 (1950).

If Y is an unknown $m\times 1$ matrix and A_i $(i=1, \cdots, n)$ are constant $m\times m$ matrices, the total differential system $dY=\sum A_i Y dx_i$ is passive (completely integrable) if $A_iA_j=A_jA_i$. It is shown that a general solution can be expressed in finite form involving polynomials and exponentials if the characteristic roots of the matrices A_i are known. The result had previously been given by N. Saltykow [C. R. Acad. Sci. Paris **225**, 520–521 (1947); these Rev. **9**, 186] and E. Vessiot [ibid. **226**, 289–291 (1948); these Rev. **9**, 354] respectively for the special cases where the roots of all the matrices A_i are simple and the roots of one matrix A_i are simple. *J. M. Thomas* (Durham, N. C.).

Popovici, Constantin. Exceptions to the rule which gives the number and continuity of the integrals of a partial differential equation with boundary conditions. Acad. Repub. Pop. Române. Bul. Şti. A. **2**, 627–630 (1950). (Romanian. Russian and French summaries)

The author shows that for given $a(x, y)$ and $b(x, y)$,

$$z(x, y)=a(x, y)f(x)+b(x, y)\varphi(y)$$

is not uniquely determined by its values on the lines $y=x$ and $y=\lambda x$ and comments on the solutions of the partial differential equation satisfied by z for arbitrary (twice differentiable) f and φ. *A. Erdélyi* (Pasadena, Calif.).

Gårding, Lars. On a lemma by H. Weyl. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **20**, 250–253 (1950).

Le théorème démontré est le suivant: Toute distribution [voir le rapporteur, Théorie des distributions, tome 1, Actualités Sci. Ind., no. 1091=Publ. Inst. Math. Univ. Strasbourg **9**, Hermann, Paris, 1950; ces Rev. **12**, 31], solution d'une équation aux dérivées partielles elliptique à coefficients indéfiniment dérivables, est une solution usuelle, elle-même une fonction indéfiniment dérivable. Ce théorème est identique à celui du rapporteur [loc. cit., p. 137], avec les mêmes hypothèses. La démonstration est la même, sauf sur un point (il remplace un passage à la limite par une relation algébrique): C'est essentiellement la démonstration très voisine donnée dans de Rham et Kodaira [Harmonic Integrals, Institute for Advanced Study, Princeton, N. J., 1950; ces Rev. **12**, 279]. *L. Schwartz* (Nancy).

Hornich, Hans. Lösbarkeit einer speziellen Differentialgleichung mit einem Parameter und Transcendenz von Zahlen. Monatsh. Math. **54**, 183–187 (1950).

The writer considers the equation

$$(1) \qquad a^2(r^2\partial^2u/\partial r^2+r\partial u/\partial r)+\partial^2u/\partial\phi^2=f(r, \phi),$$

where $f(r, \phi)=(1-r^2)^{-1}\sum_{m=0}^{\infty}e^{2m+1}\cos(2m+1)\phi$, for $0<a<1$ and $r<1$. He considers this an analogous problem to that treated by Bourgin and Duffin [Bull. Amer. Math. Soc. **45**, 851–858 (1939); these Rev. **1**, 120] in that the existence and unicity of a regular solution of (1) in the unit circle depends on the algebraic character of a. [It seems of interest to transform the author's equation to

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial\varphi^2}=f(ax, \phi)=((e^{ax}\cos\phi)(1-e^{2ax}\cos 2\phi+e^{4ax})^{-1})$$

by summing the series for $f(r, \phi)$ and replacing r by e^{ax} so that a appears now in $f(x, \phi)$ alone.] *D. G. Bourgin.*

De Donder, Th. Simplification de la méthode d'intégration d'Hadamard. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 545–547 (1950).

The simplification referred to in the title is effected by transforming the differential equation by the introduction of a new unknown function for which the Cauchy data vanish identically. The result is in a more explicit form than that given in a previous paper by the author [same Bull. (5) **30**, 763–765 (1943); these Rev. **7**, 206].

D. C. Lewis (Baltimore, Md.).

Bareiss, Erwin. Hyperbolische Systeme von partiellen Differentialgleichungen mit konstanten Koeffizienten. Comment. Math. Helv. **24**, 291–331 (1950).

A linear system of first order partial differential equations with constant coefficients can be written symbolically in the form

$$\mathbf{Lu}=\left(\sum_{\lambda=0}^{m-1}\mathbf{A}_\lambda\frac{\partial}{\partial x_\lambda}+\mathbf{B}\right)\mathbf{u}=\mathbf{f}.$$

Here the n unknown functions of the m independent variables x_λ are combined into a vector $\mathbf{u}$; $\mathbf{A}_\lambda$ and $\mathbf{B}$ are square matrices of order n and $\mathbf{f}$ a vector. With the matrix operator $\mathbf{L}$ there is associated the determinant $|L|$, which is a scalar differential operator of order n. This paper is concerned

with a special class of operators $\mathbf{L}$, which are characterized algebraically by the fact that they possess a multiplicator $\mathbf{M}$ of degree 1. This means that there exists a matrix $\mathbf{M}$, whose elements are first order operators and a scalar irreducible second order operator Γ_0, such that $\mathbf{ML}=\Gamma_0\mathbf{E}$, where $\mathbf{E}$ is the unit matrix. The existence of a multiplicator of first degree is shown to be equivalent to the conditions that $|L|=\Gamma_0^{k_n}$ and that every $(n-1)$-rowed minor of $\mathbf{L}$ is divisible by $\Gamma_0^{k_n-1}$.

With a system of this type one can associate a system of hypercomplex numbers and hypercomplex functions, which correspond to the solutions of the differential equations. For the hypercomplex functions an integral theorem analogous to Cauchy's theorem is derived. As a special case of this theorem one obtains in the elliptic case the representation of a solution in the interior of a region in terms of the Cauchy data on the boundary. The author constructs the elementary solution of the system in the totally hyperbolic case and obtains from his integral formula the solution of the Cauchy problem for data on a space-like hypersurface. Here the discussion of the singular integrals over the characteristic manifolds is analogous to the one used by Hadamard for equations of second order. *F. John.*

Hölder, Ernst. Symmetrische Behandlung des Cauchyschen Anfangswertproblems bei einer Monge-Ampèreschen Differentialgleichung vom hyperbolischen Typ. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Mat.-Nat. Kl. **95**, no. 1, 57–70 (1943).

The method of solution of H. Lewy [Math. Ann. **98**, 179–191 (1928)] as applied to the Cauchy problem for a hyperbolic Monge-Ampère equation

$$(1)\quad a^{11}+a^{22}r+2a^{12}s+a^{14}t+a^{24}(rt-s^2)=0, \qquad a^{ij}=a^{ij}(x, y, z, p, q)$$

is interpreted in terms of the geometry of lines. Putting $x_0=z$, $x_1=x$, $x_2=y$, $x_3=p$, $x_4=q$, we can reduce (1) to the condition that an alternating quadratic differential form $\sum_{i,j}a^{ij}[dx_idx_j]$ and a linear differential form $dx_0-x_3dx_1-x_4dx_2$ are to vanish simultaneously on a 2-dimensional manifold. With every element of an integral surface there is associated the complex of lines with equations

$$\sum_{i,j}a^{ij}p_{ij}=0, \qquad \sum_{i,j}b^{ij}p_{ij}=2(p_{12}+p_{34})=0$$

in Plücker coordinates p_{ij}. The focal lines of this complex give the characteristic strip elements. From this point of view the reduction of (1) by means of the characteristic strips to a system of equations that can be solved by iteration presents itself naturally. *F. John.*

Beckert, Herbert. Existenz- und Eindeutigkeitsbeweise für das Differenzenverfahren zur Lösung des Anfangswertproblems, des gemischten Anfangs-Randwert- und des charakteristischen Problems einer hyperbolischen Differentialgleichung zweiter Ordnung mit zwei unabhängigen Variablen. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **97**, no. 4, 42 pp. (1950).

The present paper is an extension and amplification of a proof for the existence and uniqueness of the solution of a quasi-linear second order partial differential equation of hyperbolic type, first given by H. Lewy [Math. Ann. **98**, 179–191 (1927), p. 186]. The method is based on the use of finite difference representation in the plane of characteristic coordinates. The author discusses in detail the case where the initial conditions are given along characteristic lines. *H. Polachek* (White Oak, Md.).

Ladyženskaya, O. A. On the uniqueness of the solution of Cauchy's problem for a linear parabolic equation. Mat. Sbornik N.S. **27(69)**, 175–184 (1950). (Russian)

A. N. Tychonoff [Rec. Math. [Mat. Sbornik] (1) **42**, 199–216 (1935)] has considered unbounded solutions of the Cauchy problem for the heat equation, and proved that: (1) if the continuous solution $U(x, t)$ of the heat equation $\partial U/\partial t=\partial^2 U/\partial x^2$ has continuous derivatives appearing in the equation on the infinite strip $0<t<t_0$, it grows so slowly that

$$\max_{0\leq t\leq t_0}|U(x, t)|\leq Ce^{Cx^2},$$

and U vanishes for $t=0$, then U vanishes throughout the strip; (2) for any $\epsilon>0$ there exists a nontrivial solution $U(x, t)$ of the heat equation, satisfying all the hypotheses of (1), save that the growth hypothesis is replaced by

$$\max_{0\leq t\leq t_0}|U(x, t)|\leq Ce^{Cx^{2+\epsilon}}.$$

I. G. Petrovskiĭ [Bull. Univ. État, Moscou, Sect. A. 1, no. 7 (1938)] established that Cauchy's problem is well posed for bounded solutions of general linear parabolic systems whose coefficients depend on t, and raised the question of extending Tychonoff's results to these systems. The present paper contains an extension of (1) to the parabolic equation

$$\frac{\partial U}{\partial t}=\sum_{\sum_{k=1}^{n}k_i=2p}a_{(k_1,\cdots,k_n)}(t)\frac{\partial^{2p}U}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}},$$

the corresponding growth condition being

$$\max_{0\leq t\leq t_0}|U(x, t)|\leq Ce^{Cr^{4p/(2p-1)}(x)},$$

where $r(x)=(\sum_{i=1}^{n}x_i^2)^{\frac{1}{2}}$ and $x=(x_1, \cdots, x_n)$; and also an extension of (2) to the parabolic equation

$$\frac{\partial U}{\partial t}=(-1)^{p-1}\frac{\partial^{2p}U}{\partial x^{2p}}.$$

J. B. Diaz (College Park, Md.).

Greco, Donato. Una nuova applicazione del metodo delle trasformate alla risoluzione di un problema al contorno per un'equazione di tipo parabolico. Giorn. Mat. Battaglini (4) **4(80)**, 102–128 (1951).

The author considers the partial differential equation $u_{xx}=u_t+p(x)u+F(x, t)$, the boundary conditions

$$\alpha_1u_t(0, t)-\alpha_1'u(0, t)-u_x(0, t)=f_1(t),$$
$$\alpha_2u_t(\pi, t)-\alpha_2'u(\pi, t)-u_x(\pi, t)=f_2(t),$$

and the initial condition $u(x, 0)=u_0(x)$. The independent variables are restricted to the rectangle R defined by $0\leq x\leq\pi$, $0\leq t\leq\delta$. The existence and uniqueness of a solution are proved under the following hypotheses: (1) $p(x)$ is non-negative, continuous, and of bounded variation in $(0, \pi)$. (2) $F(x, t)$ and $F_t(x, t)$ are continuous in R, and for every t in $(0, \delta)$ these are functions (of x) of bounded variation in $(0, \pi)$. (3) The α's are real constants, α_1 being positive and α_2 negative. (4) $f_1(t)$ and $f_2(t)$ are of class C^1 in $(0, \delta)$. (5) $u_0(x)$ is continuous and of bounded variation in $(0, \pi)$. *L. A. MacColl* (New York, N. Y.).

van Gorcum, A. H. Theoretical considerations on the conduction of fluctuating heat flow. Appl. Sci. Research A. **2**, 272–280 (1950).

The author considers the problem of heat conduction as a linear transformation of the variables T_0 and W_0, the

amplitudes of temperature and heat flow, into T_l and W_l, the amplitudes at the other surface. This transformation is written

$$T_l = P_{11}T_0 + P_{12}W_0, \qquad W_l = P_{21}T_0 + P_{22}W_0, \tag{1}$$

and the coefficients P_{ij} are derived for the cases of a flat wall of thickness l and a cylindrical wall of finite thickness. In the first instance the coefficients are trigonometric functions and in the second one Bessel functions. In each case it is shown that the determinant $|P_{ij}| = 1$. *C. G. Maple.*

Rubinštein, L. I. On the asymptotic behavior of the phase separation boundary in the one-dimensional problem of Stefan. Doklady Akad. Nauk SSSR (N.S.) **77**, 37–40 (1951). (Russian)

The author considers the asymptotic behavior of the phase separation boundary in the one-dimensional problem of Stefan for a cylinder in the case when radiation from one end of the cylinder takes place according to Newton's law. The solution of this problem for the case of small time intervals can be obtained from a system of integral equations as was shown by the author in earlier papers [same Doklady (N.S.) **58**, 217–220 (1947); these Rev. **9**, 287]. The present paper establishes a method for the extension of the solution to an arbitrarily large interval of time. The method is based on two lemmas dealing with the extremal values of subparabolic and superparabolic functions. Having proved these lemmas, the author states that the extension of the solution to large intervals of time can be made in a manner analogous to that given in one of his earlier papers [same Doklady (N.S.) **62**, 753–756 (1948); these Rev. **10**, 254]. *H. P. Thielman* (Ames, Iowa).

Dacev, A. B. On the linear problem of Stefan—the case of alternating phases. Doklady Akad. Nauk SSSR (N.S.) **75**, 631–634 (1950). (Russian)

Consider the problem of Stefan for the case in which there are several plane homogeneous layers of some substance (such as water or lead) in which the finite number of layers are alternately in solid and liquid states. If $u^k(x, t)$ is the temperature in the layer A_k $(k = 1, 2, \cdots, r)$, then u^k satisfies the equation (1) $a_k \partial^2 u^k/\partial x^2 = \partial u^k/\partial t$, where each a_k is constant. The initial conditions are given by (2) $u^k(x, t_0) = \Phi^k(x)$ $(x^{k-1} < x < x^k)$. At the points of separation of the several layers, the conditions of constant temperature are given by

$$u^1[s^1(t), t] = u^2[s^1(t), t] = u^2[s^2(t), t] = \cdots = u^r[s^{r-1}(t), t] = \phi_0, \tag{3}$$

and the end conditions by

$$u^1(x^0, t) = \phi(t) > \phi_0, \quad u^r(x^r, t) = \psi(t) > \phi_0, \quad t > t_0. \tag{3'}$$

At each interior point of separation, the condition of Stefan is already satisfied:

$$ds^k/dt = e(k_k \partial u^k/\partial x - k_{k+1}\partial u^{k+1}/\partial x)_{x=s^k(t)}, \quad k = 1, 2, \cdots, r-1, \tag{4}$$

where $e = (-1)^{k+1}/\rho\sigma$, σ is the specific heat and ρ the density. The problem is to find those functions $u^1(x, t)$, $u^2(x, t)$, $\cdots$, $u^r(x, t)$, $s^1(t)$, $s^2(t)$, $\cdots$, $s^{r-1}(t)$ which satisfy (1), (2), (3), (3'), and (4). The author uses the method of a preceding paper [same Doklady (N.S.) **74**, 445–448 (1950); Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. **45**, 321–352 (1949); these Rev. **12**, 263, 504] to give the solution of this problem. *C. G. Maple* (Washington, D. C.).

Sestini, Giorgio. Su due problemi di propagazione del calore in un solido eterogeneo con simmetria cilindrica. Rivista Mat. Univ. Parma **1**, 405–417 (1950).

In earlier papers [Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **6**(75), 47–65 (1942); Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 464–477 (1947); these Rev. **8**, 274; **9**, 439] the author stated two boundary value problems on the conduction of heat in coaxial cylinders for the differential equation (*) $U_{rr} + U_r/r = U_t$ but carried out his analysis for the equation $U_{rr} = U_t$. In the present paper his stated problems are solved for the desired equation (*). *F. G. Dressel* (Durham, N. C.).

Podolsky, Boris. A problem in heat conduction. J. Appl. Phys. **22**, 581–585 (1951).

A simple periodic point source of heat has a fixed position in an infinite conducting medium that is moving with a uniform velocity. The initial temperature and the temperature infinitely far from the source have a common constant value. Points and times are considered at which the transient temperature effects of the initial state can be neglected. From the equation of conduction a simple formula involving exponential and trigonometric functions is derived for the periodic temperatures in the medium at those points and times. The phase velocity is observed and compared with that for conduction in a stationary medium from a periodic point source. *R. V. Churchill* (Ann Arbor, Mich.).

Babbitt, J. D. On the differential equations of diffusion. Canadian J. Research. Sect. A. **28**, 449–474 (1950).

In this paper the equation governing the diffusion of gases through media in which adsorption of the diffusing gas takes place is considered. If N denotes the number of adsorbed molecules per unit volume, $\mathbf{u}$ the rate of flow and ϕ the so-called spreading pressure, the equation of continuity, $\operatorname{div}(N\mathbf{u}) = -\partial N/\partial t$, combined with the law $\mathbf{u} = -\gamma \operatorname{grad}\phi$ (γ = a constant) gives $\gamma \operatorname{div}(N \operatorname{grad}\phi) = \partial N/\partial t$. Combining with the adsorption isotherm which provides a relation of the form $\phi = f(N)$ we get $\gamma \operatorname{div}(N \operatorname{grad} f(N)) = \partial N/\partial t$. By discussing the stationary case when $N \operatorname{grad} f(N)$ should be a constant, the author concludes that the available body of experimental evidence supports his form of the diffusion equation. *S. Chandrasekhar* (Williams Bay, Wis.).

Carlson, Bengt. Neutron diffusion theory—the transport approximation. United States Atomic Energy Commission, Rep. **AECU-725**, 9 pp. (1950).

In problems of neutron diffusion involving anisotropic scattering one generally uses the formulae derived for isotropic scattering but replacing the scattering cross section σ_s by a mean value $\bar{\sigma}_s$. Thus for a law of scattering of the form $\frac{1}{2}(1+3b_1\mu)$ where b_1 is a constant, $\bar{\sigma}_s = \sigma_s(1-b_1)$. The accuracy of this approximation is tested against the exact theory which is available for this case. The comparison is made in terms of the critical radius of a bare sphere. It is found, for example, that the so-called end point method is never in error by more than 2.1 per cent and that the error is often considerably less (of the order of a fraction of a per cent). *S. Chandrasekhar* (Williams Bay, Wis.).

Gallone, S., e Salvetti, C. Metodi simbolici di calcolo relativi alla moltiplicazione dei neutroni. Nuovo Cimento (9) **7**, 482–500 (1950).

In this paper problems in the theory of neutron diffusion are considered which are similar to those treated by Wallace and LeCaine [National Research Council of Canada. Divi-

sion of Atomic Energy. Document no. 1480 (1946); these Rev. 9, 590] in their extensive compilation and described under sections III and IV of the review of their paper. But in contrast to Wallace and LeCaine, the authors of this paper solve the various boundary value problems by the operational method. Also the particular case of a cylindrical multiplying medium injected by a fast neutron point source is treated in some detail. *S. Chandrasekhar.*

Lyons, Detlof. Diffusion thermischer Neutronen. (Exakte Theorie mit Berücksichtigung der Anisotropie der Einzelstreuung.) Ann. Physik (6) 8, 156–175 (1950).

This paper deals with problems of transfer in infinite plane-parallel atmospheres under circumstances of isotropic scattering and albedo for single scattering less than one. [The author does not seem to be aware that exact solutions are known for these problems, not only for infinite and semi-infinite atmospheres but also for atmospheres of finite optical thicknesses.] *S. Chandrasekhar.*

Difference Equations, Special Functional Equations

Stöhr, Alfred. Über einige lineare partielle Differenzengleichungen mit konstanten Koeffizienten. I. Allgemeiner Teil. Math. Nachr. 3, 208–242 (1950).

Stöhr, Alfred. Über einige lineare partielle Differenzengleichungen mit konstanten Koeffizienten. II. Erstes Beispiel: Der Operator

$$\nabla\Phi(y_1, y_2)=\Phi(y_1+1, y_2)+\Phi(y_1-1, y_2)+\Phi(y_1, y_2+1)+\Phi(y_1, y_2-1)-4\kappa\Phi(y_1, y_2)$$

mit $\kappa>1$. Math. Nachr. 3, 295–315 (1950).

Stöhr, Alfred. Über einige lineare partielle Differenzengleichungen mit konstanten Koeffizienten. III. Zweites Beispiel: Der Operator

$$\nabla\Phi(y_1, y_2)=\Phi(y_1+1, y_2)+\Phi(y_1-1, y_2)+\Phi(y_1, y_2+1)+\Phi(y_1, y_2-1)-4\Phi(y_1, y_2).$$

Math. Nachr. 3, 330–357 (1950).

The first part of this paper deals with the general linear difference equation with constant coefficients in a cubic lattice:

$$(1)\quad \nabla\Phi(y_1, \cdots, y_n)=\sum_{\mu=1}^{m} a_\mu\Phi(y_1+k_1^{(\mu)}, \cdots, y_n+k_n^{(\mu)})=\psi(y_1, \cdots, y_n)$$

where the $k^{(\mu)}$ are integers and the a_μ constants. It is shown that there exists a submanifold A of the lattice, such that for given values of Φ on A there is one and only one solution of (1). Then Φ is a fundamental solution belonging to the operator ∇, if it satisfies (1) for the function ψ, which has the value 1 at the origin and vanishes everywhere else. The characteristic equation of (1) is given by

$$(2)\qquad f(x_1, \cdots, x_n)=\sum_{\mu=1}^{m} a_\mu x_1^{k_1^{(\mu)}}, \cdots, x_n^{k_n^{(\mu)}}=0.$$

The author gives a formal expression for the fundamental solution in the form of an integral over the manifold (3) $|x_1|=\cdots=|x_{n-1}|=1$ which involves the multiple valued function $x_n(x_1, \cdots, x_{n-1})$ determined from (2). This expression actually represents a fundamental solution, if $x_n\partial f/\partial x_n\neq 0$ on the manifold determined by (2), (3). The author then treats the special case, where one coefficient dominates: (4) $|a_1|+|a_2|+\cdots+|a_{m-1}|\leqq|a_m|$. This inequality implies that the maximum principle holds. There exists then at most one "distinguished" fundamental solution, which vanishes at infinity. It is proved that the integral expression gives the distinguished fundamental solution, provided $x_n\partial f/\partial k_n\neq 0$ on the manifold (3), (2).

Part II is concerned with equations of the form

$$(5)\quad \nabla\Phi(y_1, y_2)=\Phi(y_1+1, y_2)+\Phi(y_1-1, y_2)+\Phi(y_1, y_2+1)+\Phi(y_1, y_2-1)-4\kappa\Phi(y_1, y_2)=\psi$$

where the parameter κ is greater than 1. In this case the expression for the fundamental solution can be reduced to elliptic integrals. Part III deals with equation (5) for the case $\kappa=1$. Here (in contrast to the analogous situation for higher dimensions) the fundamental solution cannot be required to vanish at infinity. It is proved however that a distinguished solution can be characterized in this case within an arbitrary additive constant by the requirement to be of order $o((y_1^2+y_2^2)^{\frac{1}{2}})$. *F. John.*

Toscano, Letterio. Integrazione di una equazione alle differenze finite. Anais Fac. Ci. Porto 33, 155–167 (1948).

This paper deals with the homogeneous linear recurrent relation of order two with periodic coefficients $f(x)=a(x)f(x-1)+b(x)f(x-2)$, where $a(x)$ and $b(x)$ are given functions satisfying $a(x+m)=a(x)$, and $b(x+m)=b(x)$, and $f(0)$ and $f(1)$ are given. As the author observes, the problem was solved by d'Ocagne [J. École Polytech. Cahier 64, 151–224 (1894), see especially pp. 219–221], by reduction to a linear recurrent equation of order two with constant coefficients. The contribution of the present paper is to find a concise expression for the coefficients in that equation. The expression is in terms of determinants constructed from the functions $a(x)$, $b(x)$. *W. Strodt.*

Hahn, Wolfgang. Über die höheren Heineschen Reihen und eine einheitliche Theorie der sogenannten speziellen Funktionen. Math. Nachr. 3, 257–294 (1950).

This is a study of generalised basic hypergeometric series, or Heine series, in connection with linear q-difference equations with variable coefficients. Here q is a fixed number, $|q|<1$, and the basic operator ϑ is the q-difference operator

$$(1)\qquad \vartheta f(x)=\frac{f(qx)-f(x)}{x(q-1)}.$$

Clearly both differentiation and differencing are limiting cases of the ϑ operation. A linear q-difference equation is a functional equation of the form

$$(2)\qquad \sum_{i=0}^{n} b_i(x)\vartheta^i f(x)=\varphi(x).$$

A power series $\sum_{r=0}^{\infty}A_r x^r$ is called a Heine series if A_{r+1}/A_r is a fixed rational function $R(z)$ of $z=q^r$. Such series satisfy linear q-difference equations (2) whose coefficients $b_i(x)$ are linear in x: also $\varphi(x)=0$ in (2). Fundamental systems of solutions, and the linear substitutions connecting them, are investigated, as are also certain bilinear relations analogous to bilinear relations connecting hypergeometric series. The q-difference analogue of the Laplace transformation is used to elucidate further properties of the functions represented by Heine series. For this, see an earlier paper by the same author [Math. Nachr. 2, 340–379 (1949); these Rev. 11, 720]. In conclusion, the author explains that a systematic theory of Heine series provides a unification of the theory of special functions since all functions of the hypergeometric

type, many orthogonal polynomials, and many functions related to elliptic functions are expressible in terms of Heine series.

A. Erdélyi (Pasadena, Calif.).

Lur'e, A. I., and Fialko, G. M. On the stability of regulation in the presence of retardation in the measuring organ of the regulator. Akad. Nauk SSSR. Inženernyĭ Sbornik 4, no. 2, 109–112 (1948). (Russian)

A differential-difference equation system of the form (*) $\dot{x}(t)+ax(t)+by(t)=0$, $\dot{y}(t)+cy(t)=dx(t-\tau)$, is solved by assuming $x=X\exp wt$, $y=Y\exp wt$; eliminating the constants X and Y leads to a transcendental equation for w. If the roots w all have negative real parts the system (*) is said to be stable. Conditions on the parameters sufficient for stability in this sense are obtained; these conditions require the determination of the least positive root of an auxiliary transcendental equation. *J. G. Wendel.*

Cypkin, Ya. Z. The degree of stability of systems with retarded feedback. Avtomatika i Telemehanika 8, 145–155 (1947). (Russian)

The author investigates the dependence of the stability of feedback systems on a parameter τ which represents a constant delay in the feedback path. *E. N. Gilbert.*

Isaacs, Rufus. Iterates of fractional order. Canadian J. Math. 2, 409–416 (1950).

The author gives a new technique for constructing from a given function $g(x)$ the function $f(x)=g^{\frac{1}{2}}(x)$ satisfying $f[f(x)]=g(x)$. He calls an orbit (with respect to g) a set of values of which any two (x and y) are connected by a relation $g^m(x)=g^n(y)$ for some integers m and n [$g^m(x)$ is the mth iterate of $g(x)$]. Let us say that an element y succeeds the element x if $g(x)=y$. The author calls a cycle a set of values which succeed each other cyclically and a branch a set of values which succeed each other until entering in a cycle. A contraction of an orbit is formed by identifying subsets of its elements; one gets a curtailment by removing elements which do not succeed any other element. With these new concepts, the author proves the following theorems: 1. If $f[f(x)]=g(x)$ then every orbit with respect to f is a union of two (possibly identical) orbits with respect to g [one orbit containing an x and the other containing $f(x)$]. 2. $f(x)=g^{\frac{1}{2}}(x)$ exists if and only if g has only orbits of three kinds, of which the orbits of the first two kinds can be biuniquely united with each other into a new orbit and the orbits of the third kind can be united with themselves. 3. Two distinct orbits can be united if and only if a contraction of one of them is isomorphic to a curtailment of the other. 4. An orbit can be united with itself (in the sense of 1.) to give a new orbit if and only if it contains a cycle of n elements where $n=2k+1$ and it has only two kinds of branches: the aptly chosen curtailments of the branches of the first kind are isomorphic with the contractions of all the branches of the second kind which end with the same element, this element being the kth successor of the end of the corresponding branch of the first kind. The paper also contains remarks on the existence condition of the Abel-function $\phi(x)$ satisfying $\phi[g(x)]=\phi(x)+1$, on the Menger problem asking for the $f(x)=g^{\frac{1}{2}}(x)$ of $g(x)=a+bx$ with $b<0$, and on iterates of arbitrary fractional order.

J. Aczél (Miskolc).

Kotel'nikov, P. M. On functional equations defining trigonometric functions. Mat. v Škole 1951, no. 2, 1–12 (1951). (Russian)

Expository paper.

Integral Equations

***Mihlin, S. G. Integral'nye uravneniya i ih priloženiya k nekotorym problemam mehaniki, matematičeskoĭ fiziki i tehniki. [Integral Equations and their Applications to some Problems of Mechanics, Mathematical Physics and Engineering].** 2d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1949. 380 pp.

[The first edition appeared in 1947.] This book consists of part I, containing a general exposition of regular Fredholm equations, equations with a weak singularity, and singular equations (integrations in the sense of principal values), and of part II, relating to applications of the equations mentioned to Dirichlet-type problems and some problems of hydrodynamics and elasticity, the biharmonic equation $\Delta^2W=0$, the equation $\Delta U+k^2U=0$, and boundary value problems of Hilbert-Riemann type. The integral equations are of the second kind and of one of the following three types: (1) regular Fredholm; (2) containing a weak singularity (that is, essentially reducible to (1) by a finite number of iterations); (3) singular equations (as indicated above). For (1) the author gives the method of successive approximations, degenerate kernels, Fourier series in the general case, and the classic Fredholm theorems. Regular symmetric kernels are considered and the Hilbert-Schmidt theory is expounded; various effective methods are given for the determination of the characteristic values. Further, for type (3) most of the developments are in the complex plane; the kernels are of Cauchy or Hilbert type (of importance in the theory of analytic functions); composition of singular integrals is developed; the case of open (sufficiently smooth) contours is treated and use is made of Riemann boundary-value problems and of the Plemelj formulas. In part II, in addition to the material mentioned, the author presents Dirichlet-type problems in connection with multiply connected regions and their conformal mapping, representation of biharmonic functions, the boundary value problems for the biharmonic equation, and aerial flow around an airplane-wing. Also considered are some integrals analogous to potentials ($\Delta U+k^2U=0$) and heat potentials. The symmetric integral equations are applied to problems like that of the vibrating string or rod. Equations of type (3) are applied in connection with a variety of boundary value problems of Hilbert-Riemann type as well as to the mixed problem of elasticity. *W. J. Trjitzinsky.*

Carafa, Mario. Risoluzione in termini finiti dell'equazione integrale di Fredholm generale, nel campo analitico. Ann. Scuola Norm. Super. Pisa (3) 4, 175–190 (1950).

This paper is concerned with the solution of the linear integral equation of the second kind:

$$g(x)=f(x)+\lambda\int_a^b K(x,y)g(y)dy$$

where the functions involved are analytic in certain regions of the complex plane. The underlying methods are the same as those expounded in a previous paper [Collectanea Math.

1, 1–62 (1948); these Rev. 10, 461]; a change in the procedure gives a result of wider applicability.

T. H. Hildebrandt (Ann Arbor, Mich.).

Tranter, C. J. On some dual integral equations. Quart. J. Math., Oxford Ser. (2) 2, 60–66 (1951).

A formal solution of the dual integral equations

$$\text{(A)}\quad \begin{cases} \displaystyle\int_0^\infty t\phi(t)J_\nu(\rho t)dt = f(\rho) & (0<\rho<1), \\ \displaystyle\int_0^\infty \phi(t)J_\nu(\rho t)dt = F(\rho) & (\rho>1) \end{cases}$$

is given, where $f(\rho)$ and $F(\rho)$ are given functions and $\phi(t)$ is to be determined. The solution was known previously only for the case $F(\rho)\equiv 0$. The formal solution in the general case is

$$\phi(t) = H(t) + (\tfrac{1}{2}\pi t)^{\frac{1}{2}}\int_0^1 s^{\nu+\frac{1}{2}}\chi(s)J_{\nu+\frac{1}{2}}(st)ds,$$

where

$$H(t) = F(1)J_{\nu+1}(t) + t\int_1^\infty \rho F(\rho)J_\nu(t\rho)d\rho,$$

$$\chi(s) = \frac{2}{\pi s^{2\nu}}\int_0^s \rho^{\nu+1}\left[f(\rho) - \int_0^\infty tH(t)J_\nu(\rho t)dt\right]\frac{d\rho}{(s^2-\rho^2)^{\frac{1}{2}}}.$$

Similar formal analysis is applicable to the case when the conditions $0<\rho<1$ and $\rho>1$ in (*) are interchanged.

E. T. Copson (St. Andrews).

Tranter, C. J. On some dual integral equations occurring in potential problems with axial symmetry. Quart. J. Mech. Appl. Math. 3, 411–419 (1950).

The problem of determining the potential due to an electrified circular disc at unit potential placed with its plane parallel to and equidistant from two earthed parallel plates can be reduced to the problem of solving the dual integral equations

$$\text{(i)}\quad \int_0^\infty G(u)f(u)J_0(\rho u)du = g(\rho)\ (0<\rho<1),$$

$$\text{(ii)}\quad \int_0^\infty f(u)J_0(\rho u)du = 0\quad (\rho>1),$$

where $g(\rho)=1$, $G(u)=u^{-1}\tanh hu$, the disc being of unit radius and the plates being at a distance $2h$ apart. The limiting case $G(u)=u^{-1}$ was solved by Titchmarsh [Theory of Fourier Integrals, Oxford, 1937, p. 334] by using the Hankel transform. A formal solution is given here for general values of $g(\rho)$ and $G(u)$. It is shown that, neglecting convergence considerations, (ii) is satisfied by $f(u)=u^{1-k}\sum_{m=0}^\infty a_m J_{2m+k}(u)$ $(k>0)$. The coefficients a_m satisfy, by (i), the simultaneous equations $\sum_{m=0}^\infty a_m\int_0^\infty G(u)u^{1-2k}J_{2m+k}(u)J_{2n+k}(u)du = E(n,k)$, where $E(n,k)$ is a known function of n and k depending on $g(\rho)$. In the particular case $G(u)=u^{2k-2}$, it follows that (iii) $a_n=(4n+2k)E(n,k)$. In some applications, $G(u)$ behaves like a power of u, and it is then most convenient to choose the parameter k so that $G(u)-u^{2k-2}$ is fairly small; then a_n will differ slightly from the value (iii). The results are necessarily complicated in the general case, but the solution is completed for the disc problem, in which case the appropriate value of k is $\frac{1}{2}$, since $G(u)=(\tanh hu)/u\cong 1/u$ when h is large.

E. T. Copson (St. Andrews).

Parodi, Maurice. Sur une méthode de résolution de certaines équations intégrales dont le noyau est une fonction irrationnelle. Ann. Soc. Sci. Bruxelles. Sér. I. 64, 156–159 (1950).

The author applies the inverse Laplace transformation to the integral equation $\int_0^\infty K(x,s)f(x)dx=\theta(s)$, and hence obtains formal solutions when $K=[(s^2+x^2)^{\frac{1}{2}}-s]^{\frac{1}{2}}$, $[(s^2+x^2)^{\frac{1}{2}}-s]^\nu$, $\{[(s^4+x^4)^{\frac{1}{2}}\pm s^2]^{\frac{1}{2}}/(s^4+x^4)^{\frac{1}{2}}\}^{\frac{1}{2}}$.

A. Erdélyi (Pasadena, Calif.).

Davis, Philip. Some theorems for infinite systems of linear equations. J. Indian Math. Soc. (N.S.) 14, 139–155 (1950).

The infinite system (1) $\sum_{k=0}^\infty a_{pk}\beta_k=0$ $(p=0,1,\cdots)$ is said to possess a restriction if there exists a sequence of numbers χ_n such that $|\beta_n|\leqq\chi_n$ (all n) and $\{\beta_n\}$ satisfying (1) implies that $\beta_n\equiv 0$. The restriction is positive, semi-positive, nonzero according as, respectively, $\chi_n>0$ for all n, $\chi_n>0$ for infinitely many n, χ_n is not identically zero. The matrix $A=(a_{pk})$ is said to be of class R_1, R_2, R_3 respectively if: (R_1) there exists a submatrix, consisting of infinitely many rows of A, that can be rearranged so that all principal (diagonal) minors have nonzero determinants; (R_2) it is possible to delete a number of columns of A (zero, finitely many, or infinitely many) such that the remaining $\infty\times\infty$ matrix is of class R_1; (R_3) at least one element a_{pk} is not zero. The possession of a restriction and the classes R are related as follows: System (1) has a (i) positive, (ii) semi-positive, (iii) nonzero restriction respectively if and only if matrix A is (i) of class R_1, (ii) of class R_2, (iii) of class R_3.

The triangular (or sum-equation) system

$$\text{(2)}\quad \beta_p + \sum_{k=p+1}^\infty a_{pk}\beta_k = 0$$

is shown always to have positive restriction. Further results are given for (2) and for the corresponding nonhomogeneous system. For example, let two nonnegative functions $\psi(n)$, $\phi(n)$ exist such that (3) $|a_{pk}|\leqq\psi(p)\phi(k)$ $(k>p)$, and set $H(n)=\prod_{j=1}^n[1+\psi(j)\phi(j)]$, $R_n=R_n(n;\beta)=\sum_{j=n}^\infty|\beta_j|\phi(j)$, where $\{\beta_k\}$ satisfies (2); then $\liminf_{n\to\infty}H(n)R(n)=0$ implies that $\beta_k\equiv 0$. The above results are applied to certain linear differential operators that lead to systems of form (1) and (2).

I. M. Sheffer (State College, Pa.).

Vaĭnberg, M. M. On the characteristic elements of a class of nonlinear operators. Doklady Akad. Nauk SSSR (N.S.) 75, 609–612 (1950). (Russian)

The methods of this paper are variation-topological; the proofs are merely indicated; the terminology is that of Nemyckiĭ [Uspehi Matem. Nauk no. 1, 141–174 (1936)]. (I) In order that $F(x)$, generated by $df(x,h)=(F(x),h)$ (Fréchet differential), where $f(x)$ is w.c. (weakly continuous), be w.c. for $\|x\|\leqq a$, it is necessary and sufficient that $df(x,h)$ have in the sphere $D(\theta,a)$ a bounded remainder. Consider conditions: (1) $f(x)$ is w.c. and has for $\|x\|\leqq a$ the properties $f(-x)=f(x)\geqq 0$ and $f(x)=0$ implies $x=\theta$ (with $\|\theta\|=0$); (2) $F(x)$, generated by a differential (of f), is completely continuous for $\|x\|\leqq a$, while $\|f(x)\|=0$ implies $x=\theta$. (II) Under (1), (2) the equation $F(x)=\lambda x$ has in every sphere $\|x\|=r\leqq a$ at least a denumerable infinity of geometrically distinct characteristic elements. Other conditions: (3) The symmetric, nondegenerate kernel $K(x,y)\varepsilon L_2$ (in (x,y)) has positive characteristic values; (4) $g(u,x)$, with $|g(u,x)|\leqq M|u|$, is continuous for real u for $x\varepsilon B$ (B a bounded domain in n-space), measurable in B in x, for u fixed, $g(-u,x)=-g(u,x)$, and (for $u\neq 0$) $g(u,x)\neq 0$ al-

most everywhere. (III) Under (3), (4) the equation $\mu u(x)=\int_R K(x,y)g(u(y),y)dy$ has for $0<r\leqq a$ at least a denumerable infinity of characteristic functions in L_2; their explicit representation is given. *W. J. Trjitzinsky.*

Bethe, H. A., Tonks, L., and Hurwitz, H., Jr. Neutron penetration and slowing down at intermediate distances through medium and heavy nuclei. Physical Rev. (2) 80, 11–19 (1950).

The equation governing the Fourier-Laplace transform of $\psi(z,u,\mu)$ giving the number of collisions per unit volume, unit time and unit u is

$$(*)\quad (1-iy\mu)\phi(y,\eta,\mu)=\int d\Omega'\phi(y,\eta,\mu')g(\eta,\mu')+(1/4\pi).$$

[In (*) the integration is over all solved angles.] When $g(\eta,\mu)$ has the expansion, $(1/4\pi)\sum g_l(\eta)P_l(\mu)$, in spherical harmonics the equation becomes

$$(**)\quad (1-iy\mu)\phi(y,x,\mu)=\sum_l g_l(x)\phi_l(y,x)P_l(\mu)+(1/4\pi)$$

where instead of η a new variable x has been introduced $[\eta=\frac{1}{2}Mx-1]$. For the particular problem considered $g_0(x)=[(M+1)/M](1/2x)(1-e^{-2x})$, where M is a positive integer; there are similar expressions for the other g_l's. The homogeneous equation associated with (*) determines a characteristic root $\nu(x)$: The characteristic equation is the determinant of the system

$$\phi_l(\nu,x)=\sum_n g_n(x)A_{ln}(\nu)\phi_n(\nu,x),$$

where the coefficients,

$$A_{ln}(\nu)=\frac{1}{2}\int_{-1}^{+1}\frac{P_l(\mu)P_n(\mu)}{1-\nu\mu}d\mu,$$

can all be expressed as in terms of the function

$$f=(1/2\nu)\{\ln[(1+\nu)/(1-\nu)]\}.$$

The authors suggest that for their purposes ($M\to\infty$) the function $\nu(x)$ can be determined sufficiently accurately by retaining only the terms resulting from the first and the second order harmonics. Having determined $\nu(x)$ in this fashion, the authors proceed to the solution of the nonhomogeneous equation (**). Since in evaluating the inverse Fourier transform of ϕ, the main contribution comes from the pole at $i\nu$, the solution in the neighborhood of this value is determined from (in the "Nth approximation")

$$\phi(\nu',x,\mu)=\sum_{l=0}^{N}g_l(x)\phi_l(\nu',x)\frac{P_l(\mu)}{1-\nu'\mu}+\frac{1}{4\pi(1-\nu'\mu)}.$$

In particular if $\Phi=g_0\phi_0+(1/4\pi)$, it is shown that

$$-1/4\pi g_0\Phi=\Lambda(x,\nu')$$

where $\Lambda(x,\nu)=0$ determines in fact the characteristic root $\nu(x)$. The Laplace transform of the density function $\psi_0(z,u)$ is proportional to

$$L(z,x)=2\int_{-\infty}^{+\infty}\phi_0(y,x)e^{-iyz}dy=e^{-\nu z}/g_0^2(\partial\Lambda/\partial\nu)_\nu.$$

The quantity one is interested in is the inverse Laplace transform of $L(z,x)$ and this the authors evaluate by the method of steepest descent; and they claim that the solution obtained in this manner has a high overall accuracy. Tables of various auxiliary functions needed in the calculations are given. *S. Chandrasekhar* (Williams Bay, Wis.).

Hagihara, Yusuke. On the functional equation for the transfer of radiation. I. Jap. J. Astr. Geophysics 20, 113–138 (1943).

This paper is devoted to the consideration of integro-differential equations of the form

$$(*)\quad \frac{1}{\alpha(t)}\frac{d}{dt}\left[\alpha(t)\frac{dJ_\nu(t)}{dt}\right]=\beta(\nu)J_\nu(t)-R(\nu,t)\int_0^\infty\psi(\nu')\{J_{\nu'}(t)+V(\nu',t)\}d\nu'$$

where α,β,R,ψ and V are known functions of the arguments specified. Solutions are sought which satisfy the boundary conditions

$$J_\nu+\epsilon dJ_\nu/dt=0 \text{ for } t=t_0 \quad\text{and}\quad J_\nu+\epsilon' dJ_\nu/dt=0 \text{ for } t=t_1$$

where ϵ, ϵ', t_0 and t_1 are specified constants. When $\alpha=$constant, the general solution of (*) can be found by writing the integral on the right-hand side as a sum and then replacing the sums by integrals in the final solution. An example is provided by the equation

$$\frac{d^2J_\nu}{dt^2}=\lambda^2(\nu)J_\nu-\lambda^2(\nu)G(\nu)\int_0^\infty\psi(\nu')J_{\nu'}(t)d\nu',$$

the solution of which is found in the form

$$(**)\quad J_\nu(t)=A(\nu)\exp(-x_\nu t)+B(\nu)\exp(+x_\nu t)-\lambda^2(\nu)G(\nu)\int_0^\infty\{A(\nu')\exp(-x_{\nu'}t)+B(\nu')\exp(+x_{\nu'}t)\}\frac{\psi(\nu')}{x_{\nu'}^2-\lambda_\nu^2}d\nu'$$

where $x_\nu^2=\lambda^2(\nu)[1-G(\nu)\psi(\nu)]$; also the integral over ν' on the right-hand side of (**) is to be understood in the sense of the Cauchy principal value. In (**) $A(\nu)$ and $B(\nu)$ are functions of ν to be determined by the boundary conditions. In general the consideration of these latter boundary conditions leads to integral equations of the form

$$a(\nu)A(\nu)+b(\nu)B(\nu)=\int_0^\infty\{e(\nu,\nu')A(\nu')+f(\nu,\nu')B(\nu')\}d\nu'+p(\nu),$$

$$c(\nu)A(\nu)+d(\nu)B(\nu)=\int_0^\infty\{g(\nu,\nu')A(\nu')+h(\nu,\nu')B(\nu')\}d\nu'+q(\nu),$$

where $a(\nu)$, $b(\nu)$, etc., are known functions of the arguments specified. The author shows how the methods used in the solution of Fredholm's equation of the second kind can be applied to the solution of the foregoing equations.

For the more general case when $\alpha(t)$ in (*) is not a constant, the author shows how the solution can be similarly effected by the use of suitably defined Green's functions. *S. Chandrasekhar* (Williams Bay, Wis.).

Seidel, W., and Marshak, R. E. Upper and lower bounds for the asymptotic neutron density in Milne's problem for the sphere. Canadian J. Physics 29, 72–82 (1951).

The equation of radiative transfer is considered for an infinite homogeneous spherical medium in which no sources are present except that a net flux of amount $F/4\pi r^2$ (F is a constant) is assumed to exist in the direction $-r$; it is further assumed that at the center there is a spherical hole of

radius a. It is known that for $r\to\infty$ the asymptotic form of the density of the radiation $\psi_0(r)$ is $c(1-r/r_0)$, where c is a constant and r_0 is a function of a such that for $a\to\infty$, $r_0\to a-0.7104$. Writing $r\psi_0(r)=r+q(r)$ and considering the integral equation governing $q(r)$, the authors establish upper and lower bounds for (r_0-a). A table of these upper and lower bounds is given. *S. Chandrasekhar.*

Functional Analysis, Ergodic Theory

Smithies, F. Abstract analysis. Math. Gaz. **35**, 2–7 (1951).

Expository lecture.

Grothendieck, Alexandre. Sur la complétion du dual d'un espace vectoriel localement convexe. C. R. Acad. Sci. Paris **230**, 605–606 (1950).

Let E be a linear, locally convex, topological space; let S be a class of bounded, convex, symmetric and closed subsets of E; let E_S' (resp., $\hat{E}_S'$) be the space of linear forms which are continuous on E (resp., whose restrictions to the elements of S are continuous), endowed with the topology of uniform convergence on the elements of S; and let E_0 be the subspace generated by $\bigcup S$. If E is a Hausdorff space, then the following principal results are obtained: (A) If the elements of S are precompact, the completeness of E_S' implies that the restriction to E_0 of every $u\in\hat{E}_S'$ is continuous; and (B) If the elements of S are precompact, then E_S' is dense in $\hat{E}_S'$. (A) is readily derivable from (B). The author then observes that since the dual of E is the same for the given topology and its weak topology, and that the bounded sets are weakly precompact, the above theorems will have wide applications. With the aid of the above observation and the fact that E is identical with a topological dual of its weak dual, various corollaries result. *H. Tong.*

Dieudonné, J. Matrices semi-finies et espaces localement linéairement compacts. J. Reine Angew. Math. **188**, 162–166 (1950).

Let K be a division ring and let E and F be vector spaces which are locally linearly compact over K in the sense of Lefschetz [Algebraic Topology, Amer. Math. Soc. Colloq. Publ., v. 27, New York, 1942; these Rev. **4**, 84]. Two continous linear transformations u and v from E into F are said to be equivalent if there exist automorphisms σ and τ of E and F respectively such that $\tau u\sigma=v$. The author considers the problem of classifying linear transformations from E into F with respect to this notion of equivalence. He first reduces the problem to that of classifying the dense subspaces of a linearly compact E and shows that whenever E has $\aleph_0$ factors in its canonical decomposition then any two $\aleph_0$-dimensional dense subspaces are equivalent. This leads to a solution of the original problem whenever E and F are both such that their discrete parts are $\aleph_0$-dimensional and their linearly compact parts decompose into $\aleph_0$ factors. The case in which K is the complex field yields results of Köthe and Toeplitz [J. Reine Angew. Math. **165**, 116–127 (1931)] on the classification of semi-infinite matrices. Examples are given showing that the two countability restrictions in the reduced problem cannot be dropped without altering the result. *G. W. Mackey* (Cambridge, Mass.).

Monna, A. F. Espaces linéaires à une infinité dénombrable de coordonnées. Nederl. Akad. Wetensch., Proc. **53**, 1548–1559 = Indagationes Math. **12**, 493–504 (1950).

This paper deals with an analysis of sequence spaces $\lambda=\{x\}$, $x=(x_1,\cdots)$, $x_i\in F$, a field with a nontrivial non-Archimedean valuation, and with associated matrix transformations. The investigation is analogous with certain parts of several papers by Köthe and Toeplitz [J. Reine Angew. Math. **171**, 193–226 (1934), quoted as KT] and Köthe [Math. Ann. **114**, 99–125 (1937), referred to as K1; J. Reine Angew. Math. **178**, 193–213 (1938), referred to as K2; Math. Ann. **116**, 719–732 (1939), referred to as K3]. The author uses the definitions and notations of the above papers as we shall do in this review. Many of the proofs carry over without much change; the principal difference is that in the present paper ordinary convergence replaces absolute convergence in the above cited papers in both definitions and proofs, thus taking advantage of the stronger convergence properties of non-Archimedean valuated fields.

Various elementary propositions of KT carry over directly, such as $\lambda\subset\lambda^{**}$; $\lambda^{***}=\lambda^*$; if λ is perfect (vollkommen) and $\varphi\subset\lambda$ then λ is normal and weakly complete. This latter proposition is proved as in KT, except that in the lemma, infinite series are replaced by the corresponding sups. As in K2, weak and strong convergence and topology is introduced, based on the corresponding concepts of boundedness. It is shown that in both topologies, λ is 0-dimensional. It is stated that the k-topology of K3 can be introduced for locally compact F. As in K2, the spaces of weak and strong linear functionals are introduced, and if $\lambda\supset\varphi$, λ^* equals the former (without having to be normal), while it may fail to equal the latter. The problem of the extensions of functionals from subspaces to the whole space is attacked as in K2 by reducing it to the corresponding Banach space problem in certain quotient spaces [see, e.g., Monna, same Proc. **49**, 1134–1141 = Indagationes Math. **8**, 682–689 (1946); I. S. Cohen, ibid. **51**, 693–698 = Indagationes Math. **10**, 244–249 (1948); these Rev. **9**, 43; **10**, 48]. Matrix rings and maximal matrix rings are introduced as in KT and we have again that if λ is perfect, $\Sigma(\lambda)$ is a maximal matrix ring. It is stated that the application to the theory of infinitely many equations with infinitely many unknowns can be carried out as in K2. Finally some special cases are discussed: σ_w, $\sigma=\{(x_i)\}$, $x_i\to 0$, σ_r, and symmetric spaces λ (λ^* is then also symmetric). The following are obtained: $\sigma_w{}^*=\sigma$, $\sigma^*=\sigma_r{}^*=\sigma_w$; there are no spaces such that $\lambda=\lambda^*$; metrics can be introduced into σ, σ_r, σ_w by means of $\|x\|=\sup|x_i|$ (yielding strong convergence as previously defined). If F has the trivial valuation, λ^{**} is "konvergenzfrei" (see KT). *G. K. Kalisch.*

Sebastião e Silva, J. On the topology of analytic functional spaces. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **1**, 23–102 (1950). (Portuguese. French summary)

This paper is related to a previous one by the same author [Portugaliae Math. **9**, 1–130 (1950); these Rev. **11**, 524], but can be read independently. Here the main concern is with a critical examination of methods of topologizing spaces of analytic functions. The author essays to show that Fantappiè's concept of a space of analytic functions gives rise to various complications which make it desirable to abandon the concept. Fantappiè's space satisfies the usual closure axioms $A\subset\bar{A}$, $\bar{0}=0$, $\overline{A\cup B}=\bar{A}\cup\bar{B}$, $\bar{\bar{A}}=\bar{A}$, and the weak T_0 separation axiom. The set consisting of a single element is not closed, however, and its closure contains

infinitely many distinct elements (these elements are analytic functions, each an extension of the original element). The author's contention is that the most essential part of Fantappiè's theory is the notion of a linear analytic functional, defined on a "linear region." If C is a closed nonvoid set in the extended complex plane Ω, not all of Ω, the "linear region" (C) is the family of all single-valued analytic functions f, each f defined on some domain $D(f)$ which is an open set in Ω, containing C, with the additional requirement that $f(\infty)=0$ if $\infty \varepsilon D(f)$. The set (C) is a subset of Fantappiè's function space, but the topology of the whole space is not essential, only that of (C). However, (C) is not a true vector space, there being no unique zero element, and subtraction of elements in general not being possible. The author modifies and refounds Fantappiè's theory by starting from a space $\mathfrak{F}[C]$ (as defined in the earlier review cited above) obtained by identifying certain elements of (C) as "equivalent". Thus $\mathfrak{F}[C]$ is in a certain precise sense the quotient space of (C) with respect to an equivalence relation. The topology of $\mathfrak{F}[C]$ can either be defined directly by defining convergent sequences so that we have a Fréchet limit space of type (L) or it can be regarded as induced in the quotient space by the topology of (C) according to certain prescriptions. The stronger T_1 separation axiom holds in $\mathfrak{F}[C]$ (so that single points are closed sets), but the closure axiom $\bar{\bar{A}}=\bar{A}$ fails if C is a bounded set. Here it should be noted that in reviewing the previous paper the reviewer erroneously ascribed to the author the assertion that $\mathfrak{F}[C]$ is a Banach algebra. It is a vector ring, but as a space (L) it is not metrizable. Other aspects of the topology of $\mathfrak{F}[C]$ are considered. It is possible to define an "analytic line" in $\mathfrak{F}[C]$ in a simple and natural way, and on this definition to base a development which, the author feels, attains all the aims of the Fantappiè theory, and much more, with fewer complications. *A. E. Taylor* (Los Angeles, Calif.).

Ważewski, Tadeusz. Sur l'évaluation du domaine d'existence des fonctions implicites dans le cas des espaces abstraits. Fund. Math. **37**, 5–24 (1950).

This paper gives a minimum domain of existence for a continuous function defined implicitly by an equation $f(x, y)=c$ between variables in abstract metric spaces X and Y, in terms of what may be called "upper and lower expansion ratios" of f with respect to x and to y. The "upper expansion ratio" of f with respect to x is denoted and defined by

$$\overline{\text{all}}_x f(x, y)=\limsup_{\xi \to x} \rho(f(x, y), f(\xi, y))/\rho(x, \xi)$$

and the "lower expansion ratio" of f with respect to y by

$$\underline{\text{all}}_y f(x, y)=\liminf_{\eta \to y} \rho(f(x, y), f(x, \eta))/\rho(y, \eta)$$

where ρ denotes the distance function. When X is a normed linear space, and Y is a complete, and $f(x, y)$ is continuous for $\rho(x, a)<r$, $\rho(y, b)<R$, and satisfies $\overline{\text{all}}_x f(x, y)\leqq\alpha$, $\underline{\text{all}}_y f(x, y)\geqq\beta>0$, and when $f(x, y)=f(a, b)$ has a unique continuous solution for y, near each of its solutions, then $f(x, y)=f(a, b)$ has a unique continuous solution $y=\sigma(x)$ defined for $\rho(x, a)<\min\{r, \beta R/\alpha\}$ and satisfying there a Lipschitz condition with constant α/β. When the spaces and the function f satisfy additional conditions (including differentiability) the final result can be deduced from theorem 5 of Hildebrandt and Graves [Trans. Amer. Math. Soc. **29**, 127–153 (1927), p. 152]. The author's concepts enable him to make the proof without these restrictions. The paper includes a preliminary theorem in which a set of points in the Cartesian product $X\times Y$ replaces the function f, and corollaries on inversion of transformations.

L. M. Graves (Chicago, Ill.).

Cronin, Jane. Branch points of solutions of equations in Banach space. Trans. Amer. Math. Soc. **69**, 208–231 (1950).

L'autore considera l'equazione

$$(1) \qquad (I-C)x+S(y)+T(x, y)=\theta$$

dove x e y sono punti di uno spazio di Banach $\mathfrak{X}$, θ è l'origine di $\mathfrak{X}$, I è l'operatore identico, C un operatore lineare completamente continuo, $S(y)$ un operatore verificante una condizione di Lipschitz, $T(x, y)$ un operatore anch'esse verificante in un intorno di $x=y=\theta$ una conveniente condizione di Lipschitz e tale che $T(\theta, \theta)=\theta$. Fissato y in un intorno di θ per le soluzioni della (1) viene definita come molteplicità il grado topologico di una certa trasformazione continua a un numero finito di dimensioni collegata con la (1) [cfr. anche R. Caccioppoli, Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **24**, 258–263 (1936)].

Sia k il massimo intero ($\geqq 2$) tale che per α reale riesca $T^1(\alpha x, \theta)=\alpha^k T^1(\alpha, x)$, con $T^1(\alpha, x)$ continue, e $\mathfrak{X}_1$ la varietà lineare delle soluzioni di $(I-C)x=\theta$. Se $\mathfrak{X}_1$ è unidimensionale la predetta molteplicità è k se $\mathfrak{X}$ è lineare nel corpo complesso, è ± 1 ovvero zero, secondochè k è dispari o pari, se $\mathfrak{X}$ è lineare nel corpo reale. Se $\mathfrak{X}_1$ non è unidimensionale valgono, in ipotesi piu restrittive per T, altri risultati, assai meno significativi, che qui non è possibile precisare. L'autore pone anche in relazione la sua definizione di molteplicità con quella di grado topologico di una trasformazione completamente continua dovuta a Leray e Schauder [Ann. Sci. École Norm. Sup. (3) **51**, 45–78 (1934)] e indica qualche applicazione dei propri risultati allo studio di equazioni integrali o differenziali non lineari. *C. Miranda.*

Yood, Bertram. On fixed points for semi-groups of linear operators. Proc. Amer. Math. Soc. **2**, 225–233 (1951).

Let G be a semi-group of bounded linear operators on a normed linear space X. The author considers conditions on G which imply (I) the existence of a nonzero fixed point of the semi-group G^* of adjoints T^* of elements $T \varepsilon G$. As was remarked by N. Dunford [Bol. Soc. Mat. Mexicana **3**, 1–12 (1946); these Rev. **9**, 42], this existence may be applied to obtain the existence of a nontrivial finitely additive measure on a set, invariant under a semi-group of one-to-one transformations of the set; in the case of a group, see a paper by von Neumann [Fund. Math. **13**, 73–116 (1929)]. By the Hahn-Banach theorem, (I) is equivalent to (II): $B(G)^*=X$, where $B(G)^*$ is the closure of the linear manifold $B(G)$ generated by $A(T)=\{Tx-x \mid T \varepsilon G, x \varepsilon X\}$. The author derives variations of this condition (II), refining the general ergodic theorem due to L. Alaoglu and G. Birkhoff [Ann. of Math. (2) **41**, 293–309 (1940); these Rev. **1**, 339]. Let $D(G)=\{x \varepsilon X \mid \inf_G \|Tx\|>0\}$, $D(G)^c=$ the complement of $D(G)$ and let G_1 be the totality of convex combinations of elements in G. It is proved that (I) implies (1): $D(G_1)$ is not empty, and that (1) combined with (2): $B(G)^* \subseteq D(G_1)^c$ implies (II). The semi-group G is said to satisfy (a) if there exists a number $K>0$ such that for each $T \varepsilon G_1$ there exists a bounded linear operator U with the properties $UT \varepsilon G_1$, $\|U\| \leqq K$ and $\|UT\| \leqq K$; if U may be chosen from G_1, the condition is called (b). The condition (b) or the condition $\lim_{n \to \infty} \|T^n/n\|=0$ implies $A(G) \subseteq D(G_1)^c$, since $n^{-1}\sum_{\nu=1}^{n} T^\nu(T-I)=n^{-1}(T^{n+1}-T)$. It is proved that, under (a),

(2) is implied by (3): $A(G)\subseteq D(G_1)^*$ and $T[A(G)]\subseteq D(G_1)^*$ for $T\in G_1$. Moreover, (3) is satisfied if either (i) G is Abelian or (ii) G is a solvable group such that the derived sequence $G^{(0)}=G, G^{(1)}, G^{(2)}, \cdots$ all satisfy (b) (this last condition is surely satisfied if $\sup_n a_n\|T^n\|\leqq K$). For the case of a finite group G, (II) is proved to be equivalent to (1). Finally the results are applied to obtain an extension of von Neumann's results referred to above. *K. Yosida* (Nagoya).

Aoki, Tosio. On the stability of the linear transformation in Banach spaces. J. Math. Soc. Japan **2**, 64–66 (1950).

Let E, E' be Banach spaces and let f satisfy

$$(1)\quad (\|x\|^p+\|y\|^p)^{-1}\|f(x+y)-f(x)-f(y)\|\leqq\epsilon,\quad 0\leqq p<1;$$

then there is a unique additive transformation U such that

$$(2)\qquad \|x\|^{-p}\|f(x)-U(x)\|\leqq\epsilon.$$

The case $p=0$ was treated by Hyers [Proc. Nat. Acad. Sci. U. S. A. **27**, 222–224 (1941); these Rev. **2**, 315]. Hyers' method effects the proof. [The reviewer has shown the right sides of (1) and (2) cannot be replaced by the condition that the lim sup, as $\|x\|$ and $\|y\|\to\infty$, is 0 when $p=1$.] *D. G. Bourgin* (Urbana, Ill.).

Newburgh, J. D. A topology for closed operators. Ann. of Math. (2) **53**, 250–255 (1951).

The author discusses the topology of the set of closed operators from a Banach space X to a Banach space Y from the following point of view. Consider the space $X\times Y$ of pairs (x, y) with $\|(x, y)\|=\|x\|+\|y\|$. Any closed linear transformation on X to Y defines a closed linear manifold (c.l.m.) in $X\times Y$. Thus what is desired is a metric in the space of manifolds in $X\times Y$. First, a metric is introduced in $E_1=E-\{0\}$ where E is any Banach space: If $a, b\in E_1$, $\rho(a, b)=\inf\{\epsilon\,|\,\|a-b\|<(e^\epsilon-1)\|a\|$ and $\|a-b\|<(e^\epsilon-1)\|b\|\}$. This metric is equivalent to the norm. A neighborhood of a c.l.m. F in E is: $N_\epsilon(F)=\{a\,|\,\rho(a, b)<\epsilon$ for some $b\in F$ with $a, b\in E_1\}$. The distance from F_1 to F_2 is then $\delta(F_1, F_2)=\inf\{\epsilon\,|\,F_1-\{0\}\subset N_\epsilon(F_2)$ and $F_2-\{0\}\subset N_\epsilon(F_1)\}$. This gives a metric in the space of c.l.m.'s of E. Now let $E=X\times Y$. Let $\mathfrak{F}(E)$ be the set of c.l.m.'s F in E. $F\in\mathfrak{F}(E)$ is said to be bounded over X if it corresponds to a bounded linear operator on a subspace of X; thus there is a $K>0$ such that $(x, y)\in F$ implies $\|y\|\leqq K\|x\|$. It is proved that the set of $F\in\mathfrak{F}(E)$ which are bounded over X and whose projection on X is all of X is open in $\mathfrak{F}(E)$. Also, the norm topology on the set of subspaces $F\in\mathfrak{F}(E)$ which are bounded over X and whose projection on X is all of X (that is, the uniform topology of the associated bounded operators) is equivalent to the relative topology induced by $\mathfrak{F}(E)$. Now let $\mathfrak{T}$ be the metric space of closed operators T defined from a domain in Y into X with the metric $\delta(T, T_1)=\delta(\text{graph } T, \text{graph } T_1)$. Consider the subset of $\mathfrak{T}$ consisting of those T such that T^{-1} is bounded from all of X into Y. Then the transformation of this subset which takes T into T^{-1} is defined on an open set in $\mathfrak{T}$ and is a homeomorphism between its domain and its range. *E. R. Lorch* (New York, N. Y.).

Taylor, Angus E. Spectral theory of closed distributive operators. Acta Math. **84**, 189–224 (1951).

The operational calculus is extended to closed distributive operators T of a complex Banach space X into itself. It is specifically required that the resolvent set of T should not be empty. As the author points out, the present task could also be accomplished by the device of considering analytic functions of the bounded operator R where $R=(T-\alpha I)^{-1}$, α in the resolvent set of T. Since the spectrum of T contains $\lambda=\infty$ (in the unbounded case), the author considers functions $f(\lambda)$ analytic in a region including the spectrum of T and also $\lambda=\infty$. Then using the contour (or contours in case the region has several components) C which bounds this region, one has in analogy to the formula $f(\xi)=f(\infty)+(2\pi i)^{-1}\int_C f(\lambda)(\lambda-\xi)^{-1}d\lambda$ the operational formula $f(T)=f(\infty)I+(2\pi i)^{-1}\int_C f(\lambda)(\lambda I-T)^{-1}d\lambda$. This leads to a calculus and analysis of the spectrum similar to that for the bounded case. The author also develops by direct means a theory of polynomials in T which (in the unbounded case) do not correspond to any operators of the calculus. He obtains some new criteria for the boundedness of T. It is also shown that if T is unbounded and $p(\lambda)$ is a nontrivial polynomial, then $p(T)=0$ is impossible. *E. R. Lorch.*

Hartman, Philip, and Wintner, Aurel. Lamé coordinates in Hilbert space. Amer. J. Math. **72**, 775–786 (1950).

Let H be an Hermitian matrix, bounded in Hilbert space, K the projection of H on the hyperspace orthogonal to a fixed vector Φ of length 1, E the matrix of the linear transformation $Ex=(x, \Phi)\Phi$. Some theorems are given concerning mutual relations between the spectra of K and $H+cE$ (c real). Applications are made to differential operators, Mathieu functions and infinite systems of linear equations. *C. Miranda* (Naples).

Wielandt, Helmut. Über die Eigenwertaufgaben mit reellen diskreten Eigenwerten. Math. Nachr. **4**, 308–314 (1951).

This paper may be given an equivalent formulation as follows: Let R be a normed linear space over the complex field whose completion R' (if it is not already complete) is a Hilbert space. Suppose L is a distributive operator on R to R satisfying $(Lx, y)=(x, Ly)$, $L\neq 0$. It is further assumed that (a) if a point spectrum exists then 0 is its sole possible limit point, (b) if l^{-1} is in the point spectrum its multiplicity $M(l)$ is finite, (c) $(I-lL)^{-1}$ exists with domain R for l^{-1} not in the point spectrum, and (d) if l^{-1} is in the point spectrum there exist $M(l)$ distributive operators $\{U_i\}$ such that $U_i x=0$, $i=1, \cdots, M(l)<\infty$, implies the existence of a solution of $y-lLy=x$. The condition of interest is (d) which, in conjunction with the other restrictions, seems essentially a sort of complete continuity restriction. Indeed an easy deduction from the theorem arrived at in this paper is that L is bounded on R and admits a self adjoint completely continuous extension to R'. *D. G. Bourgin.*

Putnam, C. R. On normal operators in Hilbert space. Amer. J. Math. **73**, 357–362 (1951).

By generalizing a result of Fuglede [Proc. Nat. Acad. Sci. U. S. A. **36**, 35–40 (1950); these Rev. **11**, 371], the author obtains the following theorems: 1. If N_1 and N_2 are bounded normal operators for which there is a nonsingular bounded operator T such that $N_2=TN_1T^{-1}$, then there is a unitary operator U such that $N_2=UN_1U^*$. 2. If A is a bounded matrix and B is a bounded normal matrix and if (i) $AB-BA=C$ and (ii) $CB=BC$, then $C=0$. (A counterexample is given for the case where B is not normal.) 3. Let B be a normal operator, $B=\int s dK_s$, and A and C arbitrary such that (iii) $AB\subseteq BA+C$. If C is bounded and $\neq 0$ and $CB\subseteq BC$, then A is unbounded. Furthermore, if f is an eigenfunction of B in the domain of A, then $Cf=0$. If in addition to the condition (iii) we assume $C=\lambda I+\mu B\neq 0$, then A is unbounded and if f is an eigenfunction of B in the domain of A and if s_0 is the associated eigenvalue, then $\lambda+\mu s_0=0$.

(Theorem 2 is derived as an obvious consequence of theorem 3.) *B. Gelbaum* (Minneapolis, Minn.).

Soysal, Selma. Sur une représentation fonctionnelle d'une décomposition générale de l'unité dans l'espace de Hilbert. Bull. Tech. Univ. Istanbul **2**, no. 2, 65–86 (1949). (French. Turkish summary)

The author gives a "functional representation" for a resolution of the identity $E(\lambda)$, $0 \leqq \lambda \leqq 1$, in a separable Hilbert space $\mathfrak{H}$. Let $\lambda_1, \lambda_2, \cdots$ be the points of discontinuity of $E(\lambda)$. Let $\mathfrak{H}_i$ be the range of $[E(\lambda_i) - E(\lambda_i - 0)]$ and let $\mathfrak{H}' = [\Sigma \oplus \mathfrak{H}_i]^*$. By methods of customary scope, the space $\mathfrak{H}'$ is shown to be the sum of orthogonal subspaces $\mathfrak{M}_i$ where $\mathfrak{M}_i$ is generated by the elements $E(\lambda)\phi_i$, $\phi_i \in \mathfrak{H}'$, $(\phi_i, \phi_j) = \delta_{ij}$. Finally the space $\mathfrak{H}'$ is given a representation by means of Lebesque square integrable functions $f(\xi)$ in such a way that if $f \in \mathfrak{H}'$ corresponds to $f(\xi)$, then $E(\lambda)f$ corresponds to a function $g(\xi)$ which equals $f(\xi)$ on certain intervals and otherwise equals zero. *E. R. Lorch.*

Hamburger, Hans Ludwig. Über die Zerlegung des Hilbertschen Raumes durch vollstetige lineare Transformationen. Math. Nachr. **4**, 56–69 (1951).

The author investigates completely continuous transformations C in a Hilbert space $\mathfrak{H}$. The nonzero characteristic values of C are λ_ν and are assumed to be simple; the associated characteristic vectors are ϕ_ν. The space spanned by the ϕ_ν is denoted by $\mathfrak{S}$. Here $\mathfrak{X}$ denotes the null-space of C. For the adjoint transformation C^* one introduces analoguously the spaces $\mathfrak{S}^*$ and $\mathfrak{X}^*$. Let $\mathfrak{N} = \mathfrak{S}^{*\perp}$ and $\mathfrak{N}^* = \mathfrak{S}^\perp$. It is proved that $C(\mathfrak{N}) \subseteq \mathfrak{N}$; C is quasi-nilpotent in $\mathfrak{N}$, i.e., $\lim_{n\to\infty} \|C^n x\|^{1/n} = 0$ for $x \in \mathfrak{N}$. Also, $\mathfrak{N} \cap \mathfrak{S} = \{0\}$ if and only if the biorthogonal complement of $\{\phi_\nu\}$ in $\mathfrak{S}$ spans $\mathfrak{S}$. If $\mathfrak{N}^* \cap \mathfrak{S}^* \neq \{0\}$ and if $\mathfrak{N}^* \cap \mathfrak{S}^* \subset \mathfrak{X}^*$, then one may construct a completely continuous B with simple characteristic values such that $B(\mathfrak{N}) = \{0\}$ and $BC = CB$. The principal result to which the greater part of the paper is devoted consists in the construction of a transformation C such that $\mathfrak{N} \cap \mathfrak{S}$ is of infinite dimension and $C(\mathfrak{N} \cap \mathfrak{S}) \neq \{0\}$. Details of this construction cannot be given here. *E. R. Lorch.*

Schwartz, J. A note on the space L_p^*. Proc. Amer. Math. Soc. **2**, 270–275 (1951).

The classical result that the conjugate space of $L_p(a, b)$ is $L_q(a, b)$ where $1/p + 1/q = 1$ is extended for $p > 1$ to the case where L_p is constructed with respect to an arbitrary measure (possibly non-σ-finite). The technique involves the use of special finite-dimensional arguments and the Radon-Nikodým theorem. An example (attributed to T. Botts) is given to show the theorem is false for $p = 1$, and a modified form of the theorem is given to cover this case: The conjugate space of L_1 is the set of absolutely continuous, countably additive set functions μ (defined on the sets of finite measure) for which $|\mu| = \sup_{0<m(A)<\infty} |\mu(A)|/m(A) < \infty$ where m is the originally given measure. An analog of the classical criterion that a measure be the integral of a function in L_p concludes the paper. *B. Gelbaum.*

Cameron, R. H. A "Simpson's rule" for the numerical evaluation of Wiener's integrals in function space. Duke Math. J. **18**, 111–130 (1951).

The author develops formulae for approximating Wiener integrals by multiple Lebesgue integrals in two distinct ways. From the standpoint of numerical computation, the main result is the following: Let C be the space of all real continuous functions $x(t)$ on the interval $I = [0, 1]$ which vanish at $t = 0$, and let C' be the larger space of all real functions which vanish at $t = 0$ and are continuous on I except at a single point where they may have a finite jump. Set $\alpha_j(t) = 2^{\frac{1}{2}} \sin (j - \frac{1}{2})\pi t$,

$$\psi_n(\xi, t) = \psi_n(\xi_1, \cdots, \xi_n, t) = \frac{2}{\pi} \sum_{j=1}^{n} \frac{\xi_j \alpha_j(t)}{2j-1},$$

$$\varphi(s, t) = \begin{cases} 2^{-\frac{1}{2}} \operatorname{sgn}(s) & (0 \leqq |s| < t \leqq 1), \\ 0 & (0 \leqq t \leqq |s| \leqq 1), \end{cases}$$

and let $\varphi^n(s, t)$ be the nth partial sum in the orthogonal development of $\varphi(s, t)$, considered as a function of t, in terms of the $\alpha_j(t)$'s. Define

$$J_n(F) = \frac{1}{2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e_n(\xi) d\xi_1 \cdots d\xi_n \times \int_{-1}^{1} F[\psi_n(\xi, \cdot) + \varphi(s, \cdot) - \varphi^n(s, \cdot)] ds,$$

where $e_n(\xi) = \pi^{-\frac{1}{2}n} \exp\{-\sum_1^n \xi_j^2\}$. Then, under certain smoothness and growth hypotheses, which involve positive constants A and B ($B < \frac{1}{4}\pi^2$), on the functional F considered on the space C', $\int_{C'} F(x) d_w x = J_n(F) + \epsilon_n$, where for $n \geqq 5$, $|\epsilon_n| \leqq A n^{-2} \pi^{-2} (1 + 18/35n) \sec^{\frac{1}{2}} B^{\frac{1}{2}}$. *R. E. Graves.*

Cameron, R. H., and Graves, Ross E. Additive functionals on a space of continuous functions. I. Trans. Amer. Math. Soc. **70**, 160–176 (1951).

The authors obtain a characterization and representation for essentially additive functionals which belong to L_2 over the space C of functions $x(t)$ which are continuous over the interval $0 \leqq t \leqq 1$ and which vanish at $t = 0$. The Wiener measure on the space C is used and the phrase essentially additive functional is understood to mean a functional which is equal almost everywhere to one which is additive. They show that a functional of $L_2(C)$ is essentially additive if and only if its Fourier-Hermite development with respect to some real complete orthonormal set contains only first degree terms. Moreover, if this condition is satisfied for some real complete orthonormal set they show that it is satisfied for every such set. The necessity of the condition depends upon a fundamental lemma proved by the authors. The lemma is so presented that it applies both to additive functionals and to periodic functionals. As the authors indicate, the sufficiency of the condition follows quite readily from a theorem of Kolmogoroff [Math. Ann. **99**, 309–319 (1928)]. They give a short independent proof of the sufficiency of the condition. Their proof actually yields a little more; namely, the result that if a functional of class $L_2(C)$ is essentially additive then it is essentially linear. For the representation theorem, they prove that a functional $F(x)$ is essentially additive and of class $L_2(C)$ if and only if it has the form $F(x) = \int_0^1 f(t) \bar{d}x(t)$ for almost all x in C, where $f(t) \in L_2(0, 1)$. Other results of seemingly quite different nature are obtained; for example, they show that if M is a measurable subset of C which is a linear manifold over the field of rational numbers then the measure of M is either 0 or 1. This result follows from the portion of their fundamental lemma which relates to periodic functionals. Out of this they show that two measurable additive functionals are equal either almost everywhere or almost nowhere. The main tools used in the paper are the generalized Riemann-Stieltjes integral of Paley, Wiener, and Zygmund; and the Fourier-Hermite expansion and the translation theorem of Cameron and the reviewer. *W. T. Martin.*

Nakamura, Masahiro. Notes on Banach space. XII. A remark on a theorem of Gelfand and Neumark. Tôhoku Math. J. (2) **2**, 182–187 (1950).

Let R be a complex Banach algebra with identity element and an involution $x \to x^*$ such that $\|xx^*\| = \|x\|^2$. R is said to be "Archimedean" if $\sum_{i=1}^{n} x_i x_i^* = 0$ implies each $x_i = 0$. The author proves that, if R is Archimedean, then R is isometrically isomorphic to an algebra of operators on a Hilbert space. The proof follows methods used by I. Segal [Bull. Amer. Math. Soc. **53**, 73–88 (1947); these Rev. **8**, 520] (see also M. A. Naĭmark [Uspehi Matem. Nauk (N.S.) **3**, no. 5(27), 52–145 (1948); these Rev. **10**, 308 = Amer. Math. Soc. Translation no. **25**; these Rev. **12**, 111]). As the author points out, the condition that R be Archimedean is equivalent to the Gelfand-Naimark condition that R be symmetric ($1+xx^*$ has an inverse for every x) so that his theorem is equivalent to that of Gelfand-Naimark [Mat. Sbornik N.S. **12**(**54**), 197–213 (1943); these Rev. **5**, 147] whose paper, incidentally, was not available to the author. The author also notes that his proof is shortened by some results of I. Kaplansky [Duke Math. J. **16**, 399–418 (1949); these Rev. **11**, 115]. The Kaplansky paper was not available to the author until after his paper was submitted for publication.

C. E. Rickart (New Haven, Conn.).

Kreĭn, M. G. Hermitian-positive kernels in homogeneous spaces. II. Ukrain. Nat. Žurnal **2**, 10–59 (1950). (Russian)

[Part I, which was in vol. 1 of this journal, is not available.] Let G be a locally compact Abelian group and let $\hat{G}$ be its character group. Let $A(\hat{G})$ denote the Banach algebra of all bounded Radon measures μ on $\hat{G}$ with convolution as multiplication and $\|\mu\|$ total variation of μ. The Fourier-Stieltjes transform maps $A(\hat{G})$ isomorphically onto the algebra R_G of all functions on G which are finite linear combinations of continuous positive definite functions, multiplication in R_G being the usual pointwise multiplication of functions. If f is the correspondent in R_G of μ in $A(\hat{G})$ and $f = f_1 - f_2$ where f_1 and f_2 are positive definite then $\|\mu\| = \inf(\|f_1\| + \|f_2\|) = \inf(f_1(e) + f_2(e))$ where the inf is taken over all representations of f as such a difference. Thus an algebra R_G algebraically and topologically isomorphic to $A(\hat{G})$ may be defined without reference to $\hat{G}$ and this definition has meaning when G is neither locally compact nor Abelian. In this paper the author makes the suggested definition and studies the properties of the resulting class of Banach algebras. Actually, somewhat more than half of the paper is concerned with a still more general notion. Let H be a closed subgroup of the topological group G and let Q be the homogeneous space of left H cosets. Let R_{QG} be the linear span of the continuous positive definite functions on $Q \times Q$ which are invariant under the transformations $q, p \to sq, sp$ ($q, p \in Q$, $s \in G$). Then R_{QG} is closed with respect to multiplication and may be normed by analogy with R_G. Here $f(e)$ is replaced by the constant value of $f(p, p)$. When $H = \{e\}$, R_{QG} reduces in an obvious manner to R_G.

The following is a partial list of results. R_{QG} is always a Banach algebra. If Q is discrete then R_{QG} is a conjugate space of a space of functions in such a manner that the positive definite functions correspond to the positive linear functionals. In the definition of the norm the inf concerned is actually achieved. Let Z be the set of all elementary continuous positive definite functions on G. Then $f \in R_G$ if there exists $M > 0$ such that $|\sum_{j=1}^{n} c_j f(s_j)| \leq M \sup_{x \in z} |\sum_{j=1}^{n} c_j x(s_j)|$ for all $s_1, s_2, \cdots, s_n$ in G and all complex numbers $c_1, c_2, \cdots, c_n$. If G is locally compact, this condition is also necessary. If G is compact then R_G consists precisely of all f on G such that $f(x) = \sum_\alpha \mathrm{Tr}(A_\alpha U_\alpha(x))$ and $\sum_\alpha \|A_\alpha\|_1 < \infty$ where U_α is a generic irreducible representation of G, A_α is a matrix and $A \to \|A\|_1$ is a certain norm for matrices. If G is compact R_G is analytically complete in the sense that $\varphi(f)$ is in R_G whenever f is in R_G and φ is an analytic function whose domain includes the range of f. An expansion analogous to that above may be written down for functions on Q and the set of all functions on Q for which the expansion is valid may be made into an analytically complete Banach algebra. Let $\mathfrak{A}$ be the set of all continuous functions on Q whose transforms under G span a finite dimensional vector space. If F is a linear functional on $\mathfrak{A}$ such that $F(|h|^2) \geqq 0$ whenever $h \in \mathfrak{A}$ then $F(h) \geqq 0$ whenever h is a nonnegative element of $\mathfrak{A}$. If $\mathfrak{A}$ and $\mathfrak{A}'$ belong to Q and Q' respectively and if $\mathfrak{A}$ and $\mathfrak{A}'$ are isomorphic as *-algebras, then Q and Q' are homeomorphic.

In the next to the last section of the paper the duality between compact groups and quadratic block algebras announced earlier [Doklady Akad. Nauk SSSR (N.S.) **69**, 725–728 (1949); these Rev. **11**, 491] is worked out in detail. Many of the other ideas in this paper were announced by the author in a series of notes [C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 13–17, 18–22; **29**, 275–280, 355–359 (1940); **30**, 5–8, 9–12, 484–488 (1941); these Rev. **2**, 315, 316].

G. W. Mackey (Cambridge, Mass.).

Cotlar, Mischa, and Ricabarra, R. A. On transformations of sets and Koopman's operators. Revista Unión Mat. Argentina **14**, 232–254 (1950). (Spanish)

[Cf. the authors' previous article in the same vol., 49–63 (1949); these Rev. **12**, 85.] Let E be a set, K a σ-field of subsets of E, m a countably additive positive measure on K with $m(E) = 1$. If $h: E \to E$ is such that $h(X) \in K$ and $m\{h(X)\} = m(X)$ whenever $X \in K$ then the Koopman operator T_h associated to h is the transformation on $L^2(m)$ defined by $(T_h f)(P) = f\{h(P)\}$, $f \in L^2(m)$ and $P \in E$ (usually the consideration of T_h is restricted to the case in which $h^{-1}: E \to E$ exists and is K- and m-preserving so that T_h is unitary). A linear continuous operator T on $L^2(m)$ is said to be equivalent to a Koopman operator if there are E', K', m' and a Koopman operator T' on $L^2(m')$ such that $VT = T'V$, V being a linear homeomorphism between $L^2(m)$ and $L^2(m')$ which, when restricted to characteristic functions of measurable sets, induces a σ-isomorphism between K and K'. The authors extend a well-known fact remarked by von Neumann by showing that T is equivalent to a Koopman operator if and only if $T(fg) = T(f) \cdot T(g)$ and one of the ergodic theorems quoted by them is satisfied. A large part of this article is devoted to the study of three ergodic decompositions of E with respect to a normalized countably additive and multiplicative set function $T: K \to K$ and their applications to recent results of Dunford and Miller [Trans. Amer. Math. Soc. **60**, 538–548 (1946); these Rev. **8**, 280] and Y. N. Dowker [Duke Math. J. **14**, 1051–1061 (1947); these Rev. **9**, 359].

L. Nachbin (Rio de Janeiro).

Anzai, Hirotada. Ergodic skew product transformations on the torus. Osaka Math. J. **3**, 83–99 (1951).

Let X denote the real numbers mod 1, A the class of measurable functions from X into X. This paper studies transformations of the torus $X \times X$ of the form $T(x, y) = (x + \gamma, \alpha(x) + y)$, where γ is a fixed irrational number and $\alpha(x) \in A$. Let Ξ denote the subclass of A consisting of functions of the form $\theta(x) - \theta(x + \gamma)$, $\theta \in A$. The main results

are as follows. (1) λ is a proper value of T if and only if $p\alpha(x)-\lambda\in\Xi$ for some integer p. (2) T is ergodic if and only if $p\alpha(x)\in\Xi$ only for $p=0$. (3) The transformations corresponding to $\alpha(x)$ and $\beta(x)$, assumed ergodic, are isomorphic (i.e., equivalent under some measure preserving transformation of the torus) if and only if $\alpha(x)\pm\beta(x+u)\in\Xi$ for some $u\in X$. (4) An ergodic T has a pure point spectrum if and only if T is isomorphic to a product of two rotations. (5) The transformations T_m $(m=1, 2, \cdots)$ corresponding to $\alpha(x)=mx$ are ergodic and any two of them are spectrally isomorphic, but no two are isomorphic. (For another example, see Halmos [Bull. Amer. Math. Soc. 55, 1015–1034 (1949), especially p. 1025; these Rev. 11, 373].) (6) If $\alpha(x)=\rho\chi(x)$, where ρ is irrational and χ is the characteristic function of a subinterval E of X, then nonabsolutely continuous spectrum appears in the orthogonal complement H of the subspace of L_2 consisting of functions of x only. γ and E can be so chosen that T is ergodic and has pure continuous spectrum on H. *J. C. Oxtoby* (Bryn Mawr, Pa.).

Utz, W. R. Almost periodic geodesics on manifolds of hyperbolic type. Duke Math. J. **18**, 147–164 (1951).

Let $M(f)$ be the Riemannian manifold defined by assigning the differential metric

$$ds^2=\frac{4f^2(x)dx_i dx_i}{(1-x_ix_i)^2}$$

to the interior S_n $(n\geqq 2)$ of the unit sphere in Euclidean n-space with the assumptions that $f(x)=f(x_1, \cdots, x_n)$ is of class C^3 in S_n and there exist positive constants a and b such that $a\leqq f(P)\leqq b$ for $P\in S_n$. $M(1)$ is hyperbolic space and the geodesics or H-lines in this case are arcs of circles orthogonal to the unit sphere. If G is a properly discontinuous group of isometries of $M(1)$ such that no transformation of G other than the identity has a fixed point and it is assumed that f is invariant under G, a Riemannian manifold $M(f, G)$ can be obtained from $M(f)$ by identification of congruent points. The geodesics on $M(f, G)$ define a flow in the phase space with arclength as parameter. A geodesic g on $M(f, G)$ is almost periodic if in the phase space g returns to any neighborhood of any of its points for a relatively dense set of values of the parameter. A geodesic which is almost periodic but not periodic is strictly almost periodic (s.a.p.).

Aside from the extension to the cases under consideration of various results concerning the behavior in the large of class A geodesics, it is shown that the existence of an s.a.p. H-line on $M(1, G)$ implies the existence of a corresponding s.a.p. geodesic on $M(f, G)$. Then in the case $n=2$, it is shown by use of symbolic methods that under relatively weak hypotheses there must exist s.a.p. H-lines on $M(1, G)$ and consequently there must exist s.a.p. geodesics on $M(f, G)$. Furthermore it is shown that if G is of the first kind, the s.a.p. points of the hyperbolic flow on $M(1, G)$ are everywhere dense in the phase space. *G. A. Hedlund.*

Bernard, Richard R. Probability in dynamical transformation groups. Duke Math. J. **18**, 307–319 (1951).

This paper extends various results of G. D. Birkhoff [Dynamical Systems, Amer. Math. Soc. Colloq. Publ. no. 9, New York, 1927] and H. Hilmy [Compositio Math. 3, 227–238 (1936)] from the case of a one-parameter group to a general transformation group. Let T be a multiplicative Abelian topological group which is separable, metric, locally compact and is generated by some compact neighborhood of the identity e. Let X be a topological space and let (X, T, π) be a transformation group, i.e., π is a continuous transformation of the topological product $X\times T$ onto X such that if the image of (x, t) under π is denoted by xt, then $xe=x$ for all $x\in X$ and $(xt)s=x(ts)$ for all $x\in X$ and all $t, s\in T$. Let μ denote Haar measure on T.

If $A\subset X$, $V\subset T$, $x\in X$, let $T(x, A, V)=[t\mid t\in V, xt\in A]$. Let W be any open generator of T such that $e\in W$ and $\bar{W}$ is compact. Let K be an arbitrary translate of $\bar{W}$. If $\mu T(x, A, K^n)$ exists for all choices of K and for all positive integers n, and if

$$\lim_{n\to+\infty}\frac{\mu T(x, A, K^n)}{\mu(K^n)}$$

exists and is independent of the choice of K, this limit is called the probability that xt is in A and is denoted by $P(x, A)$. Following Hilmy, the set $A\subset X$ is defined to be a center of attraction of the point $x\in X$ provided $P(x, N)=1$ for every open set N such that $A\subset N$. The set A is a minimal center of attraction of x if A is a closed center of attraction of x and no proper subset of A has this property. A minimal center of attraction of x is invariant, and, if X is a compact metric space, corresponding to any $x\in X$ there exists a unique minimal center of attraction $A(x)$. In this case the closure of the set $\bigcup_{x\in X}A(x)=M(X)$ is called the minimal center of attraction of X. The set $M(X)$ is invariant. The main theorem of the paper states that $M(X)$ is pointwise regionally recurrent [for a definition of regional recurrence cf. Gottschalk and Hedlund, Trans. Amer. Math. Soc. 65, 348–359 (1949); these Rev. 10, 718] and hence if C denotes the center, which is the closure of the set of recurrent points of X, then $M(X)\subset C$. *G. A. Hedlund.*

Calculus of Variations

Rothe, Erich H. A relation between the type numbers of a critical point and the index of the corresponding field of gradient vectors. Math. Nachr. **4**, 12–27 (1951).

Let $I(x)=I(x_1, \cdots, x_n)$ be a real function of class C^3 neighboring the origin 0 in E^n, and having an isolated critical point there, and such that, for a neighborhood of 0, the gradient of I, at any point of the locus $I(x)=I(0)$ not 0 itself, and the vector $x-0$ are linearly independent. (This hypothesis is known to be satisfied if I is analytic. The author will show in another paper that it is true under the following hypotheses: I is of class C^{p+1}, $p\geqq 2$; all derivatives of orders less than p vanish at $x=0$; and the homogeneous form, in the Taylor's series representation, involving pth derivatives at 0, is nondegenerate in the algebraic sense.) Let j be the index of the singularity at 0 of the vector field grad $I(x)$, and m_r the rth (Morse) type number of the critical point 0 of I. The author proves the relation $j=\sum_{r=0}^{n}(-1)^r m_r$. (For definitions of terms above and below, see Alexandroff and Hopf [Topologie, Springer, Berlin, 1935] and Morse [Calculus of Variations in the Large, Amer. Math. Soc. Colloq. Publ., v. 18, New York, 1934].) The proof uses the Lefschetz fixed point formula, and the Hopf extension theorem, and is based on a discussion of the "cylindrical neighborhood" of the critical point, defined by Seifert and Threlfall [Variationsrechnung im Grossen, Hamburger Math. Einzelschr., Heft 24, Teubner, Leipzig-Berlin, 1938, p. 38]. *A. B. Brown* (Flushing, N. Y.).

Êl'sgol'c, L. Ê. Estimation of the number of critical points. Uspehi Matem. Nauk (N.S.) **5**, no. 6(40), 52–87 (1950). (Russian)

This exposition is presented as a sort of sequel to Lyusternik's "Topology and the Calculus of Variations" [Uspehi Matem. Nauk (N.S.) **1**, no. 1, 30–56 (1946); these Rev. **9**, 51]. It is devoted to the case of functions of a finite number of variables defined over compact manifolds. The paper is very full in its statement of ideas and the text contains occasional details of proofs. There are a number of very adequate drawings illustrating geometrical aspects of the theory. Chapter 1 begins with a discussion of the canonical form for a twice continuously differentiable function $f(x)$ defined on a three times differentiable manifold M^n, $x \varepsilon M^n$. It defines the type of a critical point, introduces connections between the number of critical points and various geometrical properties of M^n, studies the Morse inequalities, discusses the change of fundamental group and of Betti group of domains $f \leqq c$ on passing over critical values, and ends with the use of decompositions of M^n into a minimal number of elements (n-cells) appropriately joined for estimating the number of critical points in M^n. Chapter 2 contains a discussion of various types of category, and numerous properties of one or another category-like function, including the author's (and Frolov's) notion of length of a cycle. Chapter 3 is on the variation of the topological structure of the level surfaces $f=x$, as x passes through critical values. The results are largely contained in a paper by the author [Mat. Sbornik N.S. **23**(65), 399–418 (1948); these Rev. **10**, 392]. Chapter 4 lists many, for the most part standard, applications of critical point theory. A final section discusses several aspects of the theory not yet adequately dealt with, and promises a second paper on them. *L. Zippin.*

Gericke, Helmuth. Adjungierte Extremalflächen. Arch. Math. **2**, 273–280 (1950).

Generalizing the classical relationship between adjoint minimal surfaces, Haar [Math. Ann. **100**, 481–502 (1928)] developed a construction for an adjoint variation problem to a given variation problem, for double integral problems of a certain type, and showed that the extremal surfaces of two variation problems associated in this manner gave rise to a set of theorems analogous to those relating to adjoint minimal surfaces. Mickle [Duke Math. J. **9**, 208–227 (1942); these Rev. **4**, 49] showed subsequently that one can associate a group of six adjoint variation problems to a given problem in such a way that the analogy with adjoint minimal surfaces is preserved, at least to the same extent as in the situation studied by Haar. The present paper endeavors to generalize this line of thought still further. However, it would seem that the essential contribution of the paper is contained in the special examples of sets of adjoint extremal surfaces of variation problems associated with the area-problem in the sense of Haar and Mickle. *T. Radó.*

Mancill, Julian D. Identically non-regular problems in the calculus of variations. Univ. Nac. Tucumán. Revista A. **7**, 131–139 (1950).

This paper treats necessary conditions and sufficient conditions in order that a plane curve should minimize an integral of the form $\int P(x, y)dx+Q(x, y)dy$. The discussion includes cases when the end points are variable and when a portion of the minimizing arc lies on the boundary of the region of admissible arcs. *L. M. Graves* (Chicago, Ill.).

***Radó, Tibor. On the Problem of Plateau.** Chelsea Publishing Co., New York, N. Y., 1951. iv+109 pp. $3.25.

Photographic reproduction of a book which first appeared as Band 2, Heft 2 in the series Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer, Berlin, 1933.

Theory of Probability

Matschinski, Matthias. Sur les probabilités inverses. C. R. Acad. Sci. Paris **231**, 1282–1284 (1950).

This note attempts to justify certain uses of Bayes' formula, on the basis of two concepts and two assumptions. Concept 1 is the notion of simple event, one which cannot be regarded as the logical composition of others. Assumption 1 is that if one has no reason to consider two events as unequally probable, they must be regarded equally probable; this is applied to simple events. Concept 2 is the notion of trials not differing in any essential respect. Assumption 2 is that under such conditions, the probabilities of their various outcomes can be assigned at will, provided the same assignment is made for each trial. The logical difficulties of these concepts should be too well known at the present day to warrent comment. Assumption 1 is (as the author says) the Laplacian one. *B. O. Koopman* (New York, N. Y.).

Copeland, Arthur H., Sr. Implicative Boolean algebra. Math. Z. **53**, 285–290 (1950).

The notion of Boolean algebra is extended to involve, in addition to the usual ($\cdot$, $\vee$, $\sim$, 0, 1), a further binary operation ($\times$) defined by formal algebraic postulates. Thence is defined ($\subset$): if $y \neq 0$, then $x \subset y = z$ if and only if $x \cdot y = y \times z$. If the algebra is realized by (0, 1)-sequences in a familiar manner, the formalism is such that $x \subset y$ can be interpreted as the sequence obtained from x by the von Mises selection corresponding to y. Next, the probability $p(x)$ is defined by the usual finite measure postulates, to which is added $p(x \times y) = p(x)p(y)$; and this is shown to lead to $p(x \cdot y) = p(x)p(y \subset x)$. This makes possible the inclusion of conditional probabilities in the formalism.

B. O. Koopman (New York, N. Y.).

***Munroe, M. E. Theory of Probability.** McGraw-Hill Book Co., Inc., New York, Toronto, London, 1951. viii+213 pp. $4.50.

This is an advanced undergraduate textbook in probability which introduces the student to some of the important results of modern probability theory, including the laws of large numbers, the central limit theorem, and the law of the iterated logarithm. The level of rigor is often that of an elementary calculus textbook, except that the author is careful to point out that the proofs are incomplete. The working rules (again reminiscent of calculus textbooks) given for describing physical situations in probability terms and for computing probabilities, distributions, and expectations are a notable feature of the book. *D. Blackwell.*

Mauldon, J. G. Random division of an interval. Proc. Cambridge Philos. Soc. **47**, 331–336 (1951).

Formulae are given for the moments and distribution of the sum of the lengths of the k longest of the n intervals into which $n-1$ random points divide a unit interval.

J. L. Hodges, Jr. (Chicago, Ill.).

Levi, Beppo. On a limiting property of the sphere in n dimensions. Math. Notae **10**, 36–40 (1950). (Spanish)

A neat geometric proof of the fact discovered analytically by J. M. Hammersley [Ann. Math. Statistics **21**, 447–452 (1950); these Rev. **12**, 268] that, for large n, the distance between two points independently and uniformly distributed over the solid n-sphere of radius R is almost certain to be close to $R\sqrt{2}$. *L. J. Savage* (Paris).

Steyn, H. S. On discrete multivariate probability functions. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 23–30 (1951).

The multivariable hypergeometric series, defined by

$$F(a, b_1, \cdots, b_k, c; t_1, t_2, \cdots, t_k) = \sum \frac{(a)_x}{(c)_x}(b_1)_{x_1}\cdots(b_k)_{x_k}\frac{t_1^{x_1}}{x_1!}\cdots\frac{t_k^{x_k}}{x_k!}$$

with $(a)_x = a(a+1)\cdots(a+x-1)$, $x = x_1+x_2+\cdots+x_k$, the sum over nonnegative values of all the variables x_1 to x_k, and normalized so that $CF(a, b_1, b_2, \cdots, b_k, c; 1, 1, \cdots, 1) = 1$ is shown to have linear regression equations for each variable on all the others, and a factorial moment generating function which is also a multivariable hypergeometric series, namely $M(a_1, \cdots, a_k) = F(a, b_1, b_2, \cdots, b_k, a+b-c+1; -a_1, \cdots, -a_k)$ with $b = b_1+b_2+\cdots+b_k$. Three examples are given.

J. Riordan (New York, N. Y.).

Kanellos, S. G. On the probability of a sum of infinitely many events. Bull. Soc. Math. Grèce **25**, 104–114 (1951). (Greek. English summary)

Using the definition of probability of an event A as the limit of the ratio of the number of successes to the number of trials of an experiment to observe A, the author is able to derive the countable additivity axiom for some cases. The main theorem is that if A is the sum of a countable set of mutually exclusive events a_i of probability $P(a_i)$ then the probability of A exists and equals $\sum P(a_i)$ provided (i) the series converges and (ii) the event $R_n = a_n+a_{n+1}+\cdots$ is contained in some (finite) event b_n with a probability $P(b_n)$ tending to zero. *E. N. Gilbert* (Murray Hill, N. J.).

Kawata, Tatsuo. On the relative stability of sums of positive random variables. Kōdai Math. Sem. Rep. **1950**, 113–116 (1950).

A sequence $[X_n]$ of mutually independent random variables is said to be relatively stable with respect to a numerical sequence $[A_n]$ if $A_n^{-1}\sum_1^n X_k$ tends in probability to one. The sequence is called relatively small if X_n/A_n tends in probability to zero. If $X_n \geqq 0$ and if $F_n(x)$ is the distribution function of X_n then the author shows that a necessary and sufficient condition for relative stability and relative smallness is that for every $\tau > 0$

$$\sum_1^n \left\{1 - \int_0^\infty \exp(-x\tau/A_n)\,dF_k(x)\right\} \to \tau.$$

As an application the author derives the conditions for relative stability obtained by Bobrov [Učenye Zapiski Moskov. Gos. Univ. Matematika. **15**, 191–202 (1939)].

W. Feller (Princeton, N. J.).

Chung, K. L., and Fuchs, W. H. J. On the distribution of values of sums of random variables. Mem. Amer. Math. Soc., no. 6, 12 pp. (1951). $1.20.

Let $X_1, X_2, \cdots$ be mutually independent identically distributed k-dimensional random variables ($k = 1, 2, 3$). Let $S_n = \sum_1^n X_j$ and let $|Y|$ be the maximum of the absolute values of the components of the vector Y. The value b is called possible if for every $\epsilon > 0$, $\sup_n \Pr\{|S_n - b| < \epsilon\} > 0$, recurrent if, for every ϵ, $\Pr\{\liminf_{n\to\infty} |S_n - b| < \epsilon\} = 1$. (1) Either no value is recurrent or all possible values are recurrent. (2) There are recurrent values if and only if for $h > 0$, $\sum_1^\infty \Pr\{|S_n| < h\} = \infty$. (3) If ϕ is the characteristic function of X_1, there are recurrent values if and only if, for some $\alpha > 0$,

$$\limsup_{\rho \uparrow 1} \int_{-\alpha}^{\alpha} \frac{du}{1-\phi(u)} = \infty \qquad (k = 1).$$

If $k > 1$ the integral is replaced by the corresponding multiple integral. (4) If $k = 1$, and if $E\{X_1\}$ exists and is 0, then either every real number is recurrent or the values assumed by X_1 are (with probability 1) integral multiples of a fixed number, and all such multiples are recurrent. (5) If $k = 2$, if $E\{X_1\} = 0$, and if the components of X_1 have finite second moments, then every possible value is recurrent. (6) If $k = 3$ and if X_1 has a genuinely three-dimensional distribution, no value is recurrent. *J. L. Doob* (Urbana, Ill.).

Chung, K. L., and Kac, M. Remarks on fluctuations of sums of independent random variables. Mem. Amer. Math. Soc., no. 6, 11 pp. (1951). $1.20.

Let $X_1, X_2, \cdots$ be mutually independent random variables, each with the stable distribution having characteristic function $\exp(-|\xi|^\alpha)$. Let $S_n = \sum_1^n X_j$, let N_n be the number of changes of sign in the sequence $S_1, \cdots, S_n$, and let $M_n(a)$ be the number of S_k's with $k \leqq n$ and $|S_k| < a$. (1) If $1 < \alpha \leqq 2$, the limiting distributions of $2N_n/E\{|X_1|\}n^{1-1/\alpha}$ and $M_n(a)/2an^{1-1/\alpha}$ are shown to exist and to be the same; the limiting distribution is found explicitly. Let ϕ be the distribution with density $\exp(-x)$, $x \geqq 0$. (2) If $\alpha = 1$, the (common) limiting distribution of $\pi M_n(a)/2a \log n$ and of $N_n/(2\pi)^2(\log n)^2$ is ϕ. (3) If $\alpha < 1$, $M_n(a)$ is bounded, with probability 1, and $N_n/2D(\alpha)\log n$ has limiting distribution ϕ. The constant $D(\alpha)$ is evaluated explicitly. Chung [Ann. of Math (2) **51**, 697–706 (1950); these Rev. **11**, 731] had already evaluated the limiting distribution of $N_n/n^{\frac{1}{2}}$ under the hypothesis that the X_j's had a common distribution function with mean 0 and a finite third moment.

J. L. Doob (Urbana, Ill.).

Chung, K. L., and Erdös, P. Probability limit theorems assuming only the first moment. I. Mem. Amer. Math. Soc., no. 6, 19 pp. (1951). $1.20.

Let $X_1, X_2, \cdots$ be mutually independent integer-valued identically distributed random variables, and let $S_n = \sum_1^n X_j$. It is supposed that for every integer k, $\sup_k \Pr\{S_n = k\} > 0$, and (when so stated) that (0) $E\{X_1\}$ exists and is 0, or (∞) $E\{|X_1| \pm X_1\} = \infty$. The following are among the theorems proved. (1) $\Pr\{S_n = a\} \leqq An^{-1}$; A depends neither on n nor a. If $E\{X_1^2\} = \infty$, then $\lim_{n\to\infty} n^{\frac{1}{2}} \Pr\{S_n = a\} = 0$. (2) If X_1 has a symmetric distribution, and if $E\{|X_1|\} < \infty$, then $\lim_{n\to\infty} n \Pr\{S_n = a\} = \infty$. (3) Under (0) or ($\infty$) for each $\epsilon > 0$ and each integer a, $\Pr\{S_n = a\} \geqq (1-\epsilon)^n$ for sufficiently large n. (4) Under (0) or (∞)

$$\lim_{n\to\infty} \Pr\{S_n = a\}/\Pr\{S_{n+1} = a'\} = 1$$

and $\lim_{n\to\infty} \Pr\{S_n = a\}/\Pr\{S_n = a'\} = 1$. (5) Let T_n (T_n') be the number of values of $j \leqq n$ for which $S_j = a$ ($S_j = a'$). Then under (0) $\Pr\{\lim_{n\to\infty} T_n/T_n' = 1\} = 1$.

The corresponding discussion under stronger assumptions on the distribution of X_1 has been carried out by van

Kampen and Wintner [Amer. J. Math. 61, 965–973 (1939); these Rev. 1, 63], Esseen [Acta Math. 77, 1–125 (1945); these Rev. 7, 312] and Gnedenko [Uspehi Matem. Nauk (N.S.) 3, no. 3(25), 187–194 (1948); Doklady Akad. Nauk SSSR (N.S.) 66, 325–326 (1949); these Rev. 10, 132, 720].

J. L. Doob (Urbana, Ill.).

Donsker, Monroe D. An invariance principle for certain probability limit theorems. Mem. Amer. Math. Soc., no. 6, 12 pp. (1951). $1.20.

Let s_n be the sum of n mutually independent identically distributed random variables, each having mean 0 and standard deviation 1. Let $\{x(t), 0 \leqq t \leqq 1\}$ be the random variables of the Brownian motion stochastic process (stationary Gaussian independent increments). Let F be a function defined on the space of functions continuous on $[0, 1]$ except for at most finitely many jumps. Then $F[x(\cdot)]$ defines a random variable u_n, and on the other hand if $x_n(t, s_1, \cdots, s_n)$ is defined as s_1 for $t=0$ and as s_i for $(i-1)/n < t \leqq i/n$, $F[x_n(\cdot, s_1, \cdots, s_n)]$ defines a random variable u. It is shown that, under rather weak restrictions on F, the distribution of u_n converges to that of u, when $n \to \infty$. The asymptotic distribution of u_n is thus independent of that of the summands. Erdös and Kac [Bull. Amer. Math. Soc. 52, 292–302 (1946); 53, 1011–1020 (1947); these Rev. 7, 459; 8, 292], Fortet [Ann. Inst. H. Poincaré 11, 175–226 (1949); these Rev. 11, 731] and Mark [Bull. Amer. Math. Soc. 55, 885–900 (1949); these Rev. 11, 189] have proved this invariance property, and evaluated the asymptotic distribution in various special cases. *J. L. Doob.*

Cassels, J. W. S. An extension of the law of the iterated logarithm. Proc. Cambridge Philos. Soc. 47, 55–64 (1951).

Let $X_1, X_2, \cdots$ be a sequence of independent functions with uniform probability distribution in $0 \leqq X \leqq 1$. Denote by $F_N(\alpha, \beta)$ the number of $X_1, X_2, \cdots, X_N$ which satisfy $\alpha < X \leqq \beta$. Put $R_N(\alpha, \beta) = F_N(\alpha, \beta) - N(\beta - \alpha)$. The author proves that almost always

$$\limsup_{N \to \infty} R_N(\alpha, \beta)/N^{\frac{1}{2}}(\log \log N)^{\frac{1}{2}} = [2(\beta-\alpha)(1-\beta+\alpha)]^{\frac{1}{2}}, \tag{1}$$

$$\liminf_{N \to \infty} R_N(\alpha, \beta)/N^{\frac{1}{2}}(\log \log N)^{\frac{1}{2}} = -[2(\beta-\alpha)(1-\beta+\alpha)]^{\frac{1}{2}}.$$

For fixed α and β (1) is an easy consequence of the law of the iterated logarithm. The novelty of (1) is that it holds uniformly for all α and β. *P. Erdös* (Aberdeen).

Freudenthal, Hans. Das Petersburger Problem im Hinblick auf Grenzwertsätze der Wahrscheinlichkeitsrechnung. Math. Nachr. 4, 184–192 (1951).

The author discusses the traditional question whether or not certain sample spaces ought to be forbidden and then proceeds to prove a special case of the weak law of large numbers. He is apparently under the impression that limit theorems involving random variables without expectations have not yet been considered. *W. Feller.*

Lévy, Paul. Fonctions aléatoires $H(t)$ à valeurs entières, dépendant de processus à la fois markoviens et stationnaires. C. R. Acad. Sci. Paris 232, 1400–1402 (1951).

Without proof the author gives an outline of a rather complete new theory of Markov processes with denumerably many possible states without using the usual measurability conditions. [For examples of the possible types of degeneracy cf. the author's notes in the same C. R. 231, 467–468, 1208–1210 (1950); these Rev. 12, 269, 619]. Let $H(t)$ stand for the sample function, giving the actual state at time t. According to the author each process belongs to one of the following seven classes and all seven types actually exist. Denote by E the set of discontinuities of $H(t)$. The function $H(t)$ belongs to the class I if E has no points of accumulation in the finite; to class II if E is well-ordered and has points of accumulation; to III if E is denumerable, but not well-ordered; to IV if E is nondenumerable but of measure zero; to V if $|E|$ is positive but contains no interval; to VI if E contains at least one interval. The process belongs to type n ($n \leqq 6$) if (a) the transition probabilities $P_{h,k}(t)$ are measurable, and (b) if, for some initial condition, $H(t)$ has positive probability to belong to the class n but probability zero to belong to a higher class. The process belongs to the type VII if at least one $P_{h,k}(t)$ is not measurable. The first type is called finite, the second transfinite, the last two degenerate. The following theorems are announced. (1) The processes of type I, and only these, are determined by their infinitesimal transition probabilities. (2) The first four types are characterized by the nonexistence of instantaneous, or fictive, states [cf. the first cited review]. (3) For each of the first six types each $P_{h,k}(t)$ is either identically zero in $0 < t < \infty$ or everywhere positive. (4) For each of the first six types each $P_{h,k}(t)$ approaches a limit as $t \to \infty$. The last two theorems are false for the type VII.

Finally the author proposes the following terminology. A set G of states is called ergodic if, whenever the initial state belongs to G, the system passes with probability one infinitely often and for arbitrarily large t through each state of G. In this case $P_{h,k}(t) \to P_k$ for all $h, k \in G$, and either all P_k vanish, or all are positive. In the first case G is called weakly ergodic.

W. Feller (Princeton, N. J.).

Consael, R. Sur quelques points de la théorie des processus stochastiques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 36, 870–879 (1950).

Kolmogorov and Dmitriev [C. R. (Doklady) Acad. Sci. URSS (N.S.) 56, 5–8 (1947); these Rev. 9, 46] treated stochastic processes involving particles of n types, such that a particle of type k has at any time t probability $p_k^{(a_1, \cdots, a_n)}(t)dt$ of dividing into a_j particles of type j, $j=1, \cdots, n$, independently of the other m particles present and of the past history of the process. The author generalizes the situation by allowing immigration of particles from outside, one at a time. [Reviewer notes that this is not a real generalization since the immigration can be considered as due to the divisions of a fictitious particle of type $n+1$ which reproduces itself at each division.] The differential equations for the appropriate generating functions are then used to study in detail the birth and death process with immigration, that is the case $n=2$ under the special hypothesis that a particle of type 1 always divides into 2 particles of type 1 and that a particle of type 2 always divides into one particle of each type. He finds the generating function for the number of particles of type 1 at time t, under the assumption that the immigration rate is proportional to the birth rate, or to the death rate if the birth rate is 0.

J. L. Doob (Urbana, Ill.).

Ito, Kiyosi. On stochastic differential equations. Mem. Amer. Math. Soc., no. 4, 51 pp. (1951). $1.00.

Let $\{x_t, a \leqq t \leqq b\}$ be a family of real random variables constituting a Markov process. Kolmogorov [Math. Ann. 104, 415–458 (1931)] and (more completely) Feller [Math. Ann. 113, 113–160 (1936)] discussed processes (o) for which $\Pr\{|x_{t+h}-x_t|>\epsilon \mid x_t=\xi\}=o(h)$ for every $\epsilon>0$, and for which $E\{(x_{t+h}-x_t)^p \mid x_t=\xi\}=ha_p(t,\xi)+o(h)$, $p=1,2$, whose sample functions are almost all continuous, and processes (O) for which $\Pr\{|x_{t+h}-x_t|>\epsilon \mid x_t=\xi\}=O(h)$, whose sample functions are almost all constant except for discrete jumps. Feller also treated the combined case. These treatments centered about the partial differential equations, and more generally the integro-differential equations, satisfied by the transition probability functions $F(s,\xi;t,\eta)=\Pr\{x_t \leqq \eta \mid x_s=\xi\}$. In the present paper, the author treats a slightly wider class of Markov processes in an entirely different way. If the $[\delta_1+\delta_2]$th convolution of the η distribution

$$F(s-\delta_2, \xi, s+\delta_1, \eta+\xi)$$

converges to a limiting distribution $L_{s,\xi}$, necessarily infinitely divisible, the process is said to have the stochastic differential coefficient $L_{s,\xi}$. This coefficient exists in the cases discussed by Kolmogorov and Feller, as well as in more general cases. The author then solves the problem of finding a Markov process with assigned $L_{s,\xi}$, satisfying suitable regularity conditions. His solution is obtained by the construction of a specific stochastic process, as follows. He solves the stochastic integral equation

$$x_t(\omega)=c(\omega)+\int_a^t m[s, x_s(\omega)]ds+\int_a^t \sigma[s, x_s(\omega)]dg(s,\omega)+R.$$

Here $c(\omega)=x_a(\omega)$; m, σ are given functions derived from the Levy formula for the characteristic function of the distribution $L_{s,\xi}$; $\{g(s,\cdot), a \leqq s \leqq b\}$ is a Brownian motion process, and the second integral was defined in an earlier paper [Proc. Imp. Acad. Tokyo 20, 519–524 (1944); these Rev. 7, 313], and slightly more generally in the present one. The term R, absent in case (o), involves a new stochastic integral in the sample functions of a (fixed) infinitely divisible law, with integrand determined by $L_{s,\xi}$. The equation is solved by successive approximation, to obtain a Markov process with the desired differential coefficient, and with almost all sample functions continuous except for jumps. Finally it is shown that if the transition probability function F has a second derivative in ξ, F satisfies the usual integro-differential equation in s, ξ. Without this ad hoc hypothesis on F it is shown that F satisfies this equation in a weak sense, that is that $E\{\phi(x_t) \mid x_s=\xi\}$ satisfies a corresponding equation if ϕ is bounded and has a second derivative. The author has given a preliminary treatment of case (o) in a previous paper [Proc. Japan Acad. 22, nos. 1–4, 32–35 (1946); these Rev. 12, 191], and has extended this same case to a general differentiable manifold [Nagoya Math. J. 1, 35–47 (1950); these Rev. 12, 425]. *J. L. Doob* (Urbana, Ill.).

Itô, Seizô. On the canonical form of turbulence. Nagoya Math. J. 2, 83–92 (1951).

Let $u(t,x,\omega)$ be the deviation of the velocity of a turbulent fluid from its mean value. Here t represents the time, x is a point of 3-space, and ω is a probability parameter. If u has components u_1, u_2, u_3, define the moment tensors of turbulence by $R_{pq}(t,x;s,y)=E\{u_p(t,x,\omega)u_q(s,y,\omega)\}$. It is supposed that the turbulence is temporally and spatially homogeneous, and isotropic. K. Ito [Proc. Imp. Acad. Tokyo 20, 120–122 (1944); these Rev. 7, 313] translated the above into a rigorous mathematical form, and found the general expression for $R_{pq}(t,x;s,x)$ for any fixed x, using the known form of the covariance function of a stationary stochastic process. The author determines the general form of $R_{pq}(t,x;s,y)$, as a function of all the indicated variables using his results on positive definite functions on homogeneous spaces [Proc. Japan Acad. 26, no. 1, 17–28 (1950); these Rev. 12, 242]. The form is too complicated to restate here. *J. L. Doob* (Urbana, Ill.).

Fraser, D. A. S. Generalized hit probabilities with a Gaussian target. Ann. Math. Statistics 22, 248–255 (1951).

See the more detailed treatment given by the author previously [Generalized hit probabilities with a Gaussian target, Princeton, N. J., 1949; these Rev. 11, 257]. *J. L. Doob* (Urbana, Ill.).

Mathematical Statistics

Weibull, Martin. The distribution of the t and z variables in the case of stratified sample with individuals taken from normal parent populations with varying means. Skand. Aktuarietidskr. 33, 137–167 (1950).

A sample of N is made up of one drawing from each of N normal universes with unequal means but a common variance. The author derives the joint distribution of the sample mean and sample variance, and then that of the Student-Fisher t defined as usual. This gives the result of Robbins [Ann. Math. Statistics 19, 406–410 (1948); these Rev. 10, 134] though here an examination of the first four moments is used to provide an indication that the Student-Fisher t has a less a-normal distribution than when the population means are equal. But the present author goes further and derives the distribution of the variance ratio, here called z, obtained when the total sample sum of squares is partitioned into independent portions as in the analysis of variance. This independence is verified. Again, an examination of moment characteristics through order four suggests that this generalized z-distribution is less anormal than in the special and usual case of equal population means. Finally, the author shows that if the population means are themselves a sample from a normal universe, the resulting t- and z-distributions have the same form as if the sample came from a single normal population. *C. C. Craig* (Ann Arbor, Mich.).

Wise, M. E. The incomplete beta function as a contour integral and a quickly converging series for its inverse. Biometrika 37, 208–218 (1950).

The incomplete beta function $I_x(p,q)$ is represented as a contour integral which then is expanded in powers of $(p+\frac{1}{2}q-\frac{1}{2})^{-1}$. For given $I_x(p,q)$ and x unknown, a rapidly convergent approximation procedure for computing x in terms of percentage points of the χ^2-distribution is presented, together with an evaluation of errors. This procedure is of particular interest, since it provides the theoretical explanation and error estimates for several known approximations based on the χ^2-distribution, e.g., that of Scheffé and Tukey [Ann. Math. Statistics 15, 217 (1944); these Rev. 6, 9]. *Z. W. Birnbaum* (Seattle, Wash.).

Aroian, Leo A. On the levels of significance of the incomplete beta function and the F-distributions. Biometrika 37, 219–223 (1950).

Approximate formulae are given for computing percentage points of the incomplete beta function and of the F-distribution. Only the knowledge of percentage points of the normal distribution is needed to perform the computations for the incomplete beta function; for the F-distribution with degrees of freedom n_1, $n_2 \geqq 24$ the proposed computations require no auxiliary tables. Numerical comparisons with other known approximations and exact values are presented.

Z. W. Birnbaum (Seattle, Wash.).

Tukey, John W. Some sampling simplified. J. Amer. Statist. Assoc. 45, 501–519 (1950).

The author derives moments and sampling moments of finite populations in terms of some new symbols which simplify the many troublesome formulae associated with this problem. The reviewer, however, favors Cornfield's device [same J. 39, 236–239 (1944); these Rev. 6, 91] for obtaining these formulae. *A. M. Mood.*

Malmquist, Sten. On a property of order statistics from a rectangular distribution. Skand. Aktuarietidskr. 33, 214–222 (1950).

Let $x_1 < x_2 < \cdots < x_n$ be an ordered sample of n observations on a variate uniformly distributed on $0 < x < 1$. It is shown that $p = x_i/x_j$ with $i < j$ has a Beta distribution and is distributed independently of x_j. In particular, $p_1 = x_1/x_2$, $p_2 = x_2/x_3$, $\cdots$, $p_r = x_r/x_{r+1}$ are independent in the probability sense and are uniformly distributed on $0 < p < 1$.

A. M. Mood (Santa Monica, Calif.).

Cochran, W. G. Testing a linear relation among variances. Biometrics 7, 17–32 (1951).

Given k independent estimates v_i of true variances θ_i, where $n_i v_i/\theta_i = \chi_i^2$ with n_i degrees of freedom. The author considers the problem of testing the null hypothesis

$$\sum_{i=1}^{r} \theta_i = \sum_{i'=r+1}^{k} \theta_{i'}.$$

An approximate F-test is considered: $F' = \sum v_i / \sum v_{i'}$, with degrees of freedom

$$f_1 = (\sum v_i)^2/\sum(v_i^2/n_i); \qquad f_2 = (\sum v_{i'})^2/\sum(v_{i'}^2/n_{i'}).$$

A numerical example is given for $k = 4$, $r = 2$.

The distribution of F' is investigated for $k = 3$, $r = 1$. The true significance probabilities of F' are computed at the apparent 5% level for $n_1 = n_2 = 6$ and $n_1 = 3$, 6; $n_1 = n_2 = 12$, 24 and $n_1 = 6$, 12; $U = \theta_1/\theta_2 = 1$, 2, 4, 16. The maximum discrepancy in the Type I error was .009. The power of the test was computed (using the apparent 5% significance level) for $n_1 = n_2 = n_3 = 6$; $U = 1$, 2, 4, 16; $\varphi = \theta_3/(\theta_1 + \theta_2) = 2$, 4, 8. These results seem to indicate that F' is quite satisfactory when only three variances are involved. Work needs to be done on the important case of $k = 4$, $r = 2$. An alternative statistic $F'' = v_1/(v_2 + v_3 - v_4)$ has often been used in this case. A comparison of the Type I error and power of F' and F'' is needed. F'' has the advantage of requiring the computation of only f_2, since $f_1 = n_1$. Two other tests are compared with F': a normal deviate test and a likelihood ratio test. *R. L. Anderson* (Raleigh, N. C.).

Theil, H. A rank-invariant method of linear and polynomial regression analysis. III. Nederl. Akad. Wetensch., Proc. 53, 1397–1412 = Indagationes Math. 12, 467–482 (1950).

[For parts I and II see the same Proc. 53, 386–392, 521–525 = Indagationes Math. 12, 85–91, 173–177 (1950); these Rev. 12, 117.] The general incomplete method developed in part I is adapted and applied to finding confidence intervals and regions for the parameters in certain special cases of polynomial regression equations and of systems of regression equations. An application is also made to a multivariate prediction problem and in the concluding section, the author points out that other applications are possible, giving an illustration. *C. C. Craig* (Ann Arbor, Mich.).

Moran, P. A. P. Recent developments in ranking theory. J. Roy. Statist. Soc. Ser. B. 12, 153–162 (1950).

The author reviews recent work on rank correlation and derives (i) the expected values of Spearman's and Kendall's rank correlation coefficients and of Kendall's coefficient of consistency in paired comparisons under certain assumptions; (ii) the cumulants of Kendall's rank correlation coefficient when all rank permutations are equally probable.

W. Hoeffding (Chapel Hill, N. C.).

Daniels, H. E. Rank correlation and population models. J. Roy. Statist. Soc. Ser. B. 12, 171–181 (1950).

(I) It is shown that $|3nT - 2(n+1)R| \leqq n-2$, where T and R are Kendall's and Spearman's rank correlation coefficients in a sample of n observations. (II) Let $y_1, \cdots, y_n$ have the frequency function $\prod_{j=1}^{n} f(y_j - c - j\theta)$. The problem is considered of estimating θ when only the ranks of the y_j are given. *W. Hoeffding* (Chapel Hill, N. C.).

Davis, R. C. On minimum variance in nonregular estimation. Ann. Math. Statistics 22, 43–57 (1951).

The author considers the problem of estimating θ when x has a density $P(x)Q(\theta)e^{\theta h(x)}h[a(\theta), x, b(\theta)]$, where $h(x, y, z) = 1$ when $x \leqq y \leqq z$ and 0 elsewhere. If $a(\theta)$ is a constant a, the nth order statistic $x^{(n)}$ is a sufficient statistic from a sample of size n if and only if $h(x)$ can be chosen as zero. In this case the unique function of $x^{(n)}$ which is an unbiased estimate of $\varphi = [Q(0)]^{-1}$ is $(1 + 1/n)\int_a^{x^{(n)}} P(t)dt$. A differential equation is given, whose solution is the unique function of $x^{(n)}$ which is an unbiased estimate of θ. Analogous results are obtained when both a and b are functions of θ: the pair $(x^{(1)}, x^{(n)})$ is sufficient and, under certain regularity hypotheses, there is a unique unbiased estimate of θ which is a function of $x^{(1)}$ and $x^{(n)}$. Under certain hypotheses it is claimed that no single sufficient statistic $T(x^{(1)}, x^{(n)})$ exists; the proof given requires some regularity hypothesis on T, since any one-to-one mapping T of the plane into the real line does constitute a sufficient statistic. *D. Blackwell.*

Hammersley, J. M. On estimating restricted parameters. J. Roy. Statist. Soc. Ser. B. 12, 192–229; discussion, 230–240 (1950).

Several estimation problems are considered where the parameter space is a discrete set of points. These problems introduce variations from those usually encountered and at the same time focus attention on some of the deeper aspects of the whole general question of estimation. The extensive discussion, in particular, raises several questions in this direction. The specific problems considered and solved in whole or in part include: estimation of the integral unknown mean of a normal distribution, comparison of the

estimator [$\bar{x}+\frac{1}{2}$] with the estimator [median$+\frac{1}{2}$], estimation of the integral unknown mean of a Poisson distribution, a "random alms" type of problem, and the estimation of the molecular weight of insulin. The Cramér-Rao inequality for the variance of an estimate [see, for example, H. Cramér, Mathematical Methods of Statistics, Princeton Univ. Press, 1946; these Rev. 8, 39] is extended to discrete parameter spaces. It appears to the reviewer that the argument showing $\bar{x}$ is not sufficient for μ (the integral unknown mean of a normal distribution) does not hold if the usual definition of sufficiency is used [e.g., Lehmann and Scheffé, Sankhyā 10, 305–340 (1950); these Rev. 12, 511].

D. G. Chapman (Seattle, Wash.).

Lehmann, E. L. Consistency and unbiasedness of certain nonparametric tests. Ann. Math. Statistics 22, 165–179 (1951).

It is shown that there exist strictly unbiased and consistent tests (based on ranks) for the univariate and multivariate two- and k-sample problem, for the hypothesis of independence, and for the hypothesis of symmetry with respect to a given point. For certain hypotheses an unbiased test is necessarily a randomized one. These unbiased tests do not seem to be very efficient with respect to "most" alternatives. The two-sample test of Mann and Whitney is shown to be unbiased against the alternatives that the two cumulative distribution functions satisfy the inequality $F(t)>G(t)$ for all t. The results concerning unbiasedness remain true for discontinuous distributions if tied observations are ranked at random. Certain new (not unbiased) two-sample tests are discussed which are designed to discriminate against specified alternatives. The large-sample power of these tests and the Mann-Whitney test are obtained by means of a theorem of the reviewer [same Ann. 19, 293–325 (1948); these Rev. 10, 134].

W. Hoeffding (Chapel Hill, N. C.).

van Dantzig, D. On the consistency and the power of Wilcoxon's two sample test. Nederl. Akad. Wetensch. Proc. Ser. A. 54 = Indagationes Math. 13, 1–8 (1951).

Wilcoxon's one sided test for testing the hypothesis H that two samples $X_1, \cdots, X_m$; $Y_1, \cdots, Y_n$ with cumulative distribution functions F and G, respectively, come from the same population, rejects H if too many of the mn differences x_i-y_j are negative. Mann and Whitney [Ann. Math. Statistics 18, 50–60 (1947); these Rev. 9, 151] proved this test consistent for the case that the Y's are stochastically larger than the X's. The author points out that consistency holds against the wider class of alternatives for which $P(Y<X)<\frac{1}{2}$ and gives certain lower bounds for the power of the test.

E. L. Lehmann (Berkeley, Calif.).

***Whittle, Peter. Hypothesis Testing in Time Series Analysis.** Thesis, Uppsala University, 1951. 121 pp.

The author studies the testing of the adequacy of various types of models as the origin of a given sample $x_1, \cdots, x_n$. For the most part his tests, and the distributions of the test functions, are obtained from plausibility considerations allowing errors which can be expected to be of order of magnitude $n^{-\frac{1}{2}}$. The main emphasis is on the study of discrimination between various types of models, done in such a way that any parameter θ (which may be multidimensional) involved in a model does not appear in the test. For example H_0, H_1 may be the hypotheses that the x_j process is Gaussian, with zero means, and (respectively) a process of moving averages of order 1 or autoregressive of order 1. The basic method used involves the assumption that each parameter θ has an "a priori frequency function" h. The usual manipulations of conditional probabilities involving the θ distribution are made, but according to the author "the fact that θ has a frequency function does not at all imply that it is a random variable". Much of the work is justified by the statement that variations in h produce effects which are at most of order $O(n^{-1})$. In general his method is to obtain a likelihood function, under the hypothesis H_i, which is independent of θ, and the critical test region is then determined by the likelihood ratio. In order to obtain the likelihood function either h is given some convenient form or h may be dropped (that is, made identically constant) and θ replaced by its maximum likelihood estimate. As an example of his results, let H_0, H_1 be as above. The test function is the quotient of two quadratic forms in the x_j's. These forms are assumed to be distributed normally, and to find the distribution of their quotient it is thus necessary only to find first and second moments. This methodology is employed in many problems, including periodogram tests, and some tests are checked, with satisfactory results, on artificial data.

J. L. Doob (Urbana, Ill.).

Ogawara, Masami. A note on the test of serial correlation coefficients. Ann. Math. Statistics 22, 115–118 (1951).

Normal regression theory is used to estimate the parameters in a stationary Gaussian simple Markov process, under the assumption that alternate values of the process are fixed. Confidence limits are set up for the autocorrelation coefficient. The theory for a process of order k is also indicated.

R. L. Anderson (Raleigh, N. C.).

Banerjee, K. S. Weighing designs. Calcutta Statist. Assoc. Bull. 3, 64–76 (1950).

Expository paper. *A. M. Mood.*

Banerjee, K. S. Some contributions to Hotelling's weighing designs. Sankhyā 10, 371–382 (1950).

The author derives some designs which will estimate individual weights as well as the combined weight of all objects with equal precision. Also, some special methods are presented for maximizing determinants associated with the weighing problem.

A. M. Mood (Santa Monica, Calif.).

Williams, E. J. Experimental designs balanced for pairs of residual effects. Australian J. Sci. Research. Ser. A. 3, 351–363 (1950).

The author derives some crossover designs, that is, designs for experiments in which each of k treatments must be applied in succession to each experimental unit. The designs are balanced so that each treatment is immediately preceded the same number of times by every other ordered pair of treatments. Designs are found for $k=3, 5, 6$. An illustrative analysis of experimental data is included.

A. M. Mood (Santa Monica, Calif.).

Post, R. F., and Schiff, L. I. Statistical limitations on the resolving time of a scintillation counter. Physical Rev. (2) **80**, 1113 (1950).

Estimates related to the Poisson distribution when the time parameter t is replaced by an operational time $\tau=f(t)$.

W. Feller (Princeton, N. J.).

Mandelbrot, Benoit. Adaptation du message à la ligne de transmission. I. Quanta d'information. C. R. Acad. Sci. Paris **232**, 1638–1640 (1951).

Walsh, John E. On the large sample distribution of mortality rates based on statistically independent lives. Soc. Actuar. Trans. **2**$_1$, no. 4, 228–232; discussion, 233–234 (1950).

In mortality statistics one may save time and money by studying large samples instead of the whole population. The statistical qualities of such samples are compared with those of the underlying population.

P. Johansen (Copenhagen).

TOPOLOGY

***Favard, J. Espace et dimension.** Éditions Albin Michel, Paris, 1950. 302 pp.

This is a strictly elementary textbook intended to bring to the general mathematical reader an introduction to the topological concept of a "space" and the related rigorous definition of dimensionality. The first chapter, which occupies more than a quarter of the book, brings in the background material which the author considers necessary for this general reader. Many things are put into this text which the American reader would expect to find in other places. For example, the author starts with the positive integers and builds up the real number system, the step from the rationals to the reals being made with a rather clumsy use of nested intervals. Euclidean space of n dimensions is briefly described from a more or less intuitive viewpoint, and the chapter closes with a short discussion of simplexes and polyhedra.

The second chapter is a more formal study of set theory. Open sets are defined in n-dimensional Euclidean space using the usual idea of spherical neighborhood. Countable sets and those having at most the power of the continuum are described. The usual elementary theorems proved in a course in functions of a real variable are given here. Continuous mappings are defined and the standard mapping of a closed interval onto a square plus its interior is given to prevent the general reader from becoming overconfident. Chapter III brings in the concept of a topological space based on the usual axioms. More axioms are quickly introduced as they are needed. Continuous functions are discussed more thoroughly, and Borel sets are introduced. This chapter gives a good deal of the usual material found in introductory topology, and closes with a proof of the celebrated theorem of Hahn and Mazurkiewicz.

The last two chapters comprise eighty-three pages and give the formal study of dimension theory. The usual definition of dimensionality is shown to be equivalent to that of Lebesgue in the case where the space is compact. Euclidean space of n-dimensions is again considered and proved to have dimension n. It is shown that each compact n-dimensional set may be embedded topologically in a Euclidean space of dimension $(2n+1)$. Essential mappings are considered, homotopy is introduced, and the Jordan curve theorem is given for n dimensions. The book closes with a description of the construction of a universal one dimensional set embedded in the ordinary Euclidean three-space.

D. W. Hall (College Park, Md.).

Sikorski, Roman. Remarks on some topological spaces of high power. Fund. Math. **37**, 125–136 (1950).

The spaces $X(\omega_\mu)$ considered satisfy Kuratowskian closure axioms: I. There is an ordinal ω_μ such that if X_ξ, $\xi<\alpha<\omega_\mu$, is a sequence of sets then $\overline{\sum X_\xi}=\sum \bar{X}_\xi$. II. If X is finite then $\bar{X}=X$. III. For all X, $\bar{\bar{X}}=\bar{X}$. If $\mu=0$ these are equivalent to the classical axioms. If $\mu>0$ it is required that ω_μ be a regular ordinal. The theory developed has elements in common with that of C. Goffman and the reviewer [Trans. Amer. Math. Soc. **66**, 65–74 (1949); these Rev. **11**, 44]. It is proved that if $\mu>0$ and $X(\omega_\mu)$ is regular then it is 0-dimensional. A basis for $X(\omega_\mu)$ is a sequence G_ξ, $\xi<\omega_\mu$, of open sets such that every open set is the union of a subsequence of the G_ξ. An extension of the Urysohn metrization theorem is stated: If $X(\omega_\mu)$ is regular and has a basis then it is metrizable in an ordered field. (Necessary and sufficient conditions that a topological space be metrizable in an ordered group have been established by the authors mentioned above [Proc. Amer. Math. Soc. **1**, 750–753 (1950); these Rev. **12**, 434].) It is stated that every compact metrizable $X(\omega_\mu)$ is complete and totally bounded. The concepts here are obtained from the classical ones by replacing ω_0 by ω_μ and "finite" by "$<\aleph_\mu$." An example is given to show that the converse of this theorem is false if $\mu>0$. In this connection several problems are stated on the relation between compactness and the accessibility of the cardinal $\aleph_\mu$. The author concludes with a generalization of a theorem of Stone on Boolean algebras [Trans. Amer. Math. Soc. **41**, 375–481 (1937)]. A Boolean algebra K is called ω_μ-complete if, for every $C\subset K$ where $\aleph(C)<\aleph_\mu$, the union of all elements $A\in C$ is in K. A space $X(\omega_\mu)$ is called ω_μ-bicompact if every open covering $[G_\tau]$ of $X(\omega_\mu)$ contains a subcovering G_{τ_ξ}, $\xi<\alpha<\omega_\mu$. It is shown that: An ω_μ-complete Boolean algebra K is isomorphic with the field of subsets of an ω_μ-bicompact regular $X(\omega_\mu)$ which are both open and closed if and only if every ω_μ-additive proper ideal of K is contained in an ω_μ-additive prime ideal of K.

L. W. Cohen.

Myers, S. B. Functional uniformities. Proc. Amer. Math. Soc. **2**, 153–158 (1951).

Let $\mathfrak{F}$ be any family of equicontinuous sets F of real-valued functions f on a topological space X such that for each neighborhood V of a point x and each positive real number d there is an F in $\mathfrak{F}$ such that $d_F(x, y)\leqq d$ implies $y\in V$, where $d_F(x, y)$ is the least upper bound for all f in F of $|f(x)-f(y)|$. Let (F/d) denote the class of those (x, y) for which $d_F(x, y)<d$. The sets (F/d) for all F in $\mathfrak{F}$ and all positive d form a subbase for a uniform structure in X: the functional uniformity determined by $\mathfrak{F}$; and conversely, as is also shown, every uniform structure can be so obtained. The point of interest is that familiar families $\mathfrak{F}$ determine familiar types of uniform structures. Thus, the family of all equicontinuous sets determines the largest ("finest") uniform structure; and if each F is finite and each f bounded then and only then $\mathfrak{F}'$ leads to a compact completion, etc. One of several results of this correspondence is that every uniform structure u includes another v having the same class

of uniformly continuous real-valued functions as u and which leads to a compact completion X^- from which by relativization one recovers all real-valued functions uniformly continuous relative to u. The present approach to uniform structures seems closely related to that involving families of écarts [Bourbaki, . . . Topologie générale, Chapitre IX, Actualités Sci. Ind., no. 1045, Hermann, Paris, 1948; these Rev. 10, 260] or pseudo-metrics [Tukey, Convergence and Uniformity in Topology, Princeton University Press, 1940; these Rev. 2, 67]. Although the author does not employ these latter concepts (the functions d_F being introduced in this review merely to elucidate the relation) his primary observation is essentially that a pseudo-metric r coincides with d_F where F is the equicontinuous set of all r_x where $r_x(y)=r(x, y)$. *R. Arens* (Los Angeles, Calif.).

Cooper, J. L. B. Topologies in rings of sets. Proc. London Math. Soc. (2) **52**, 220–240 (1951).

This paper generalizes to completely additive Boolean rings the natural topology induced by Lebesgue measure on the space of Lebesgue measurable sets. Let R be such a Boolean ring. A mesh on R is a filter with basis β such that if $B\in\beta$, $x\in B$, and $y\subset x$, then $y\in B$. Given a family γ of completely additive measure functions on R, let J be a subset of the family, and let $B(J, \epsilon)=\{x\in R \mid F(x)<\epsilon \text{ for all } F\in J\}$. Then the filter of all subsets of R which contain at least one $B(J, \epsilon)$ is a mesh, and is called a regular mesh (generated by γ). The regular meshes on R form a complete lattice under the natural partial ordering, the largest mesh M being generated by the family of all measure functions on R and the smallest mesh m by the identically zero measure. Two meshes m and m' are called mutually singular if $m\wedge m'=m$; a measure function is singular with respect to a mesh m if it generates a mesh singular to m; two measure functions are mutually singular if they generate mutually singular meshes. A measure function is absolutely continuous in a mesh m if for every $\epsilon>0$ there is a set of the mesh on which the function is in absolute value less than ϵ. It is shown that any regular mesh can be put in the form $m=\bigvee m(L_\alpha)$ where $\{L_\alpha\}$ is a set of measure functions all of which are absolutely continuous with respect to m and such that any two of them are mutually singular. Also, if m is a regular mesh and L is a measure function, there is a decomposition $L=L_c+S$ where L_c is absolutely continuous in m and S is singular in m. It is then shown that the regular meshes form a Boolean algebra, if the complement of m is defined as the regular mesh generated by the set of all measure functions singular in m.

If the members of a regular mesh are taken to be the set of neighborhoods of the zero element in R, a topology is induced on R making it a topological ring. This topology in general satisfies no separation axioms; however, if R is reduced modulo its ring of null elements the resulting ring has a completely regular Hausdorff topology, which will be metric and complete if the mesh was generated by a single measure function. *S. B. Myers* (Ann Arbor, Mich.).

Freudenthal, H. Kompaktisierungen und Bikompaktisierungen. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 184–192 (1951).

The author proves the theorem that a necessary and sufficient condition that a Hausdorff space R can be imbedded in a bicompact Hausdorff space R^* with $\dim R^*/R=0$, is that R be semibicompact. If, moreover, R satisfies the 2nd countability axiom, then bicompactness may be replaced by compactness. The space R is semibicompact if a basis exists for neighborhoods of R all neighborhoods of which have bicompact boundaries. The author remarks that all the essential ideas for the proof of this theorem are implicit in his paper on "ends" [Ann. of Math. (2) **43**, 261–279 (1942); these Rev. **3**, 315]. The connection of this paper with the earlier one lies in the fact that in the presence of certain further conditions on the space R, the points of R^*/R are uniquely defined, and are the "ends" of the space R.
L. Zippin (Flushing, N. Y.).

Fort, M. K., Jr. A note on pointwise convergence. Proc. Amer. Math. Soc. **2**, 34–35 (1951).

Let t be a topology for $X=C([0, 1], R)$. Then denote by conseq (t) the class of convergent sequences in X. The author shows that if t is metrizable then conseq $(t)\neq$ conseq (p) where p is the topology of pointwise convergence. Actually, his proof shows also that conseq $(t)\neq$ conseq (p) if t provides a countable basis for any function in X. *R. Arens.*

Hahn, S. W. Universal spaces under strong homeomorphisms. Trans. Amer. Math. Soc. **70**, 301–311 (1951).

Let E_n denote a Euclidean space of n dimensions and M_n^j the subset of E_n consisting of the totality of all points with at most j rational coordinates. If A and B are subsets of E_n, then A is called strongly homeomorphic to B if there exists a homeomorphism $h(E_n)=E_n$ such that $h(A)=B$. A subset X of E_n is said to be universal under strong homeomorphisms for k-dimensional subsets of E_n $(0\leqq k\leqq n)$ provided: (1) $\dim X=k$; (2) every k-dimensional subset Y of E_n is strongly homeomorphic to a subset of X. Among the results obtained in this paper are the following: (A) For $k=n$ and for $k=n-1$, M_n^k contains a strongly homeomorphic image of every subset Y of E_n for which $\dim Y\leqq k$. (B) If $0\leqq k\leqq n-2$, then there exists no subset of E_n which is universal under strong homeomorphisms for k-dimensional subsets of E_n. *D. W. Hall* (College Park, Md.).

Please read Errata p. 1139, Math. Rev. V.13, the note to this review.

De Giorgi, Ennio. Un criterio generale di compattezza per lo spazio delle successioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **9**, 238–242 (1950).

For each sequence $x=\{x^i\}$ of real numbers and each positive integer n, let $\pi_n(x)$ be the sequence $y=\{y^i\}$ such that $y^k=x^k$ for all $k\leqq n$ and $y^k=0$ for all $k>n$. Let Σ be a collection of sequences such that for each n, $\pi_n(\Sigma)\subset\Sigma$. Let $|x-y|$ be a distance-function on Σ and $X\subset\Sigma$. For each n, let $\varphi(n \mid X)$ be the supremum of distances $|x_1-x_2|$ for x_i's in X with $|\pi_n(x_1)-\pi_n(x_2)|<1/n$. Consider the following statements: (I) $|x-y|\geqq|\pi_n(x)-\pi_n(y)|$ (all $n>0$, $x, y\in\Sigma$); (II) $\lim_{n\to\infty}|\pi_n(x)-\pi_n(y)|=|x-y|$; (III′) for each bounded sequence $\{x_i\}$ of points of Σ there is a point $x\in\Sigma$ such that for each n, $\{\pi_n(x_i)\}$ has a subsequence which converges to $\pi_n(x)$; (α) X is conditionally compact; (β) X is bounded and $\lim_{n\to\infty}\varphi(n \mid X)=0$. The author proves that [(I)&(III′)&(β)] implies (α) and that [(I)&(II)&(α)] implies (β). Thus, if (I), (II), and (III′) all hold, (β) provides a characterization of compactness. (Actually, rather than (III′) the author uses an equivalent statement concerning sequences of sets.)

His proofs are unnecessarily complicated and hypotheses excessive. For suppose (III′) and (β) hold, $\{x_i\}$ is a sequence in X, and x is a point of Σ associated with $\{x_i\}$ by (III′). (One can assume x is not one of the x_i's.) For each $\epsilon>0$ there is by (β) an n such that $\varphi(n \mid X)<\epsilon$, and since $\{\pi_n(x_i)\}$ has a subsequence converging to $\pi_n(x)$ there is an x_j such that $|\pi_n(x_j)-\pi_n(x)|<1/n$. Then by the definition of φ, $|x-x_j|<\epsilon$. Hence $\{x_i\}$ has a subsequence converging to

$x \in \Sigma$ and (α) holds. Thus $[(\text{III}')\&(\beta)]$ implies (α). On the other hand, (I) implies that for each $x, y \in \Sigma$, the sequence $|\pi_i(x)-\pi_i(y)|$ is nondecreasing. Thus if (α) holds and (β) fails there are convergent sequences $x_i \to x$ and $y_i \to y$ in X and an $\epsilon>0$ such that $|x_n-y_n|>\epsilon$ and $|\pi_m(x_n)-\pi_m(y_n)|<1/n$ whenever $m \leqq n$. By (I), π_m is continuous, so it follows that $|\pi_m(x)-\pi_m(y)|=0$ for each m, and hence $x=y$. But this contradicts the fact that $|x-y| \geqq \epsilon$, so $[(\text{I})\&(\alpha)]$ implies (β). Thus (II) is entirely unnecessary. Also, no use is made of the triangle inequality and the fact that distinct points have positive distance is needed only in the following form: if $|\pi_m(x)-\pi_m(y)|=0$ for each m, then $|x-y|=0$. Hence the assumption that $|x-y|$ is a distance-function can be greatly relaxed. *V. L. Klee, Jr.*

Sierpiński, W. Sur une définition des espaces complets. Ganita 1, 13–16 (1950).

If M denotes a metric space, $p \in M$ and $r>0$, the set $S(p, r)$ of points x of M with $px \leqq r$ is the closed sphere of center p and radius r. Then M has the property of Ascoli provided $\prod_{i=1}^{\infty} S_i \neq 0$ for each monotone decreasing sequence of closed spheres S_i with $r_i \to 0$. It is shown in this note that a metric space is complete if and only if it has the property of Ascoli. An example of a complete metric space with a monotone decreasing sequence of closed spheres having a null product shows that the condition $r_i \to 0$ cannot be suppressed. It is dispensable in complete metric spaces M such that $x, y \in M$ and $r>0$ imply the existence of $z \in M$ such that $xy+yz=xz$ and $yz=r$. *L. M. Blumenthal* (Los Angeles, Calif.).

Fulton, Lewis M., Jr. Decompositions induced under finite-to-one closed mappings. Duke Math. J. 18, 287–295 (1951).

A decomposition of a separable metric space into a finite number of closed sets $\{F_i\}$ is regular if the dimension of the intersection of each j of the sets is at most $n-j+1$. A (continuous) mapping f of a space X onto a space Y is k-to-one if each set $f^{-1}(y)$, $y \in Y$, has at most k points. Theorem. Let $f(X)=Y$ be closed and k-to-one, where X and Y are separable metric spaces, dim $X=n$. Then if $\{U_i\}$ is a finite open covering of X, there is a regular decomposition of X into sets F_i, each in some U_j, and such that the intersection of each s of the sets F_i' of the decomposition of Y induced by f is of dimension at most $n+k-s$. Corollary. If dim $Y=n+k-1$, the maximum possible, the induced decomposition is regular.

A related result, generalizing a lemma of L. V. Keldyš [Doklady Akad. Nauk SSSR (N.S.) 58, 181–184 (1947); these Rev. 9, 298], by removing compactness conditions, is the following. Let $f(X)=Y$ be a closed mapping, where X, Y are separable metric spaces, dim $X=n$. Let $\{F_i\}$ be a decomposition of Y of (Lebesgue) order m, and let U_i be an open set in Y containing F_i. Then there is a regular decomposition of X which induces in Y a decomposition of order m, each set of which is contained in some U_i.

G. S. Young (Ann Arbor, Mich.).

Cohen, Herman J. Some results concerning homogeneous plane continua. Duke Math. J. 18, 467–474 (1951).

It was shown by Mazurkiewicz [Fund. Math. 5, 137–146 (1924)] that every bounded, homogeneous, locally connected plane continuum M is a simple closed curve. The author shows that a bounded plane continuum M is a simple closed curve if (1) M is homogeneous and arcwise connected, (2) M is homogeneous and contains a simple closed curve, or (3) M is locally homogeneous and locally connected. (A space M is locally homogeneous if for each $x, y \in M$ there is a homeomorphism which throws x onto y and throws a neighborhood of x onto a neighborhood of y.) It is shown also that every bounded plane continuum which is the sum of a collection of disjoint simple closed curves is homeomorphic to a closed plane annulus. *E. E. Moise.*

Kuratowski, Casimir. On a topological problem connected with the Cantor-Bernstein theorem. Fund. Math. 37, 213–216 (1950).

The author defines two closed and bounded sets A and B, each of which is a 1-dimensional subset of the Euclidean plane. These sets are not homeomorphic although each of them is homeomorphic to a relatively open subset of the other. The problem raised by Sikorski [Colloquium Math. 1, 242 (1948)] of defining two 0-dimensional sets A and B satisfying the same conditions remains open.

D. W. Hall (College Park, Md.).

Uehara, Hiroshi. A group of automorphisms of the homotopy groups. Nagoya Math. J. 2, 73–82 (1951).

Let X be a polytope, $x_0 \in X$ a fixed point, and Ω the set of all $f: (X, x_0) \to (X, x_0)$. Let $f \in \Omega^+ \subset \Omega$ if there exists a $\varphi \in \Omega$ with $f\varphi \simeq \varphi f \simeq$ identity, the homotopy taking place in Ω. Two elements of Ω^+ are equivalent if they are homotopic in Ω. Defining $f \cdot g = g(f(x))$, the equivalence classes of Ω^+ with this operation form a group $\Sigma(X)$ which is evidently a group of automorphisms of $\pi_n(x)$ $n \geqq 1$: $A(\alpha)=f\alpha$, $f \in A$. The author constructs a group extension $\mathfrak{A}(X)$ of $\pi_1(X)$ by $\Sigma(X)$, described algebraically as follows: elements: all couples (A, ξ), $A \in \Sigma$, $\xi \in \pi_1(X)$; composition: $(A, \xi)(B, \eta)=(A \cdot B, B\xi \cdot \eta)$; operation on $\pi_n(X)$, $n \geqq 1$: $(A, \xi)(\alpha)=(A(\alpha))^{\xi}$, the classical operation on π_1 on π_n. A geometrical approach is given, realizing $\mathfrak{A}(X)$ in spaces with the proper homotopy extension properties. *J. Dugundji* (Princeton, N. J.).

Smirnov, Yu. M. On irreducible cyclic bicompacta. Uspehi Matem. Nauk (N.S.) 5, no. 6(40), 157–158 (1950). (Russian)

This note contains a proof of the following generalization to bicompact spaces of a classical theorem: Every n-dimensional irreducible cyclic bicompactum is a Cantorian manifold. Another formulation, explicit in the proof, is as follows: Let F be an n-dimensional bicompact space which can be expressed as the sum of two closed subsets F and F', such that the intersection $F \cap F'$ is at most $(n-2)$-dimensional. Then no mapping of F into the sphere S^n can be essential. A principal idea in the proof is credited to Hurewicz and Wallman [Dimension Theory, Princeton Univ. Press, 1941; these Rev. 3, 312; the reference is to a Russian translation].

L. Zippin (Flushing, N. Y.).

Koseki, Ken-iti. Über die Homöomorphien der offenen Flächen. I. Mem. Coll. Sci. Univ. Kyoto Ser. A. 26, 75–94 (1950).

This paper gives the author's arguments preliminary to obtaining a condition for the homeomorphism of two open surfaces. His condition, to be expressed in terms of mod 2 homology, is claimed to be simpler than that of Kerékjártó [Vorlesungen über Topologie. I., Springer, Berlin, 1923]. It is shown that each element of the one-dimensional homology group mod 2 (Zusammenhangsgruppe) of an open orientable surface may be represented by a simple closed polygon on a subdivision of a given triangulation of the surface. Moreover, the question as to whether the polygon separates the surface depends only on the choice of element, and not on

the choice of representing polygon. Elements whose representing polygons do separate the surface ("elements of the first kind") form a subgroup of the mod 2 homology group.
P. J. Hilton (Manchester).

Walker, Geoffrey. Sur la fibration des variétés riemanniennes. C. R. Acad. Sci. Paris **232**, 1465–1467 (1951).

A Riemannian manifold M^n, which admits a parallel field of r-planes ($0<r<n$) is known to be locally a product of an r-dimensional and an s-dimensional Riemannian manifold. Let M^n be complete, and denote by $R(x)$ and $S(x)$ the (complete) manifolds through a point x, obtained by "analytic continuation" of the local factors. Under postulates of increasing strength on the $R(x)$ and $S(x)$ it is shown that M^n is fibered by the $R(x)$, with a base space B, which has any $S(x_0)$ for covering space, that M^n is isometric with $R\times S/G$, where G is a discontinuous group of isometries of R and S ($=R(x_0)$ and $S(x_0)$), derived from the structure group of the fibering, and eventually that M^n has a double fibering by the $R(x)$ and by the $S(x)$, with base spaces A and B, such that $A\times B=M^n/G$. The first postulate states that on one of the $S(x)$ each point has a neighborhood which meets each $R(x)$ in only one point, the next says that any element of the structure group with a fix point is the identity, the third says that $R(x)\cap S(x)\cap V_{x_0}$ contains at most one point, for a suitable neighborhood V_{x_0} of x_0 and all x, for any x_0.
H. Samelson (Ann Arbor, Mich.).

Blanchfield, R. C., and Fox, R. H. Invariants of self-linking. Ann. of Math. (2) **53**, 556–564 (1951).

The authors introduce a modification of the Seifert invariants [S.-B. Preuss. Akad. Wiss. **26/29**, 811–828 (1933)] of a "group with linkage" (generalization of the one-dimensional torsion group of an orientable three-dimensional manifold). A precise relationship is given which enables the new invariants to be expressed in terms of the Seifert invariants, and conversely. However, the new system has the advantage of being defined in invariant fashion. A method for calculating the invariants of a $(4N+3)$-dimensional closed orientable manifold direct from the boundary relations and intersection numbers is indicated, and an application of this technique is promised in a forthcoming paper by one of the authors. By dualizing to cohomology, the algebraical theory may be applied to an arbitrary complex, when the linkage-multiplication has to be interpreted as a multiplication over the torsion subgroup of H^{n+1} with values in $H^{2n+1}(R)$, where R is the group of rationals mod 1. On page 562, line 6, "2N-dimensional chains" should read "$(2N+2)$-dimensional chains".
P. J. Hilton.

Blankinship, William A. Generalization of a construction of Antoine. Ann. of Math. (2) **53**, 276–297 (1951).

A set A in R^n ($n\geqq 3$) is constructed which is compact and 0-dimensional, but has a complement that is not simply connected. Like Antoine's original example, A is the intersection of a decreasing sequence of polyhedra A_r. The groups of the complements of the A_r are computed (as was first done for the case $n=3$ by R. P. Coelho [Portugaliae Math. **6**, 57–65 (1947); these Rev. **9**, 455]), and $\pi_1(S^n-A)$ obtained as a direct limit. That this group is not trivial is proved by exhibiting a representation in the symmetric group of order 6. It is further shewn [generalising J. W. Alexander, Proc. Nat. Acad. Sci. U.S.A. **10**, 10–12 (1924)] that if $1\leqq q\leqq n$, R^n contains a q-cell whose complement is not simply connected. On the negative side, it is shewn that if C is contained in a compact subset of Hilbert space H then $H-C$ is contractible to a point; and if K, in R^n ($n\geqq 3$), is compact and has a 0-dimensional projection onto some R^{n-1}, then R^n-K is simply connected.
M. H. A. Newman (Manchester).

Belck, Hans-Boris. Reguläre Faktoren von Graphen. J. Reine Angew. Math. **188**, 228–252 (1950).

The degree of a vertex x of a graph is the number of edges having x as an end, edges whose two ends coincide in x being counted twice. The graph is regular of degree g if the degree of each vertex is g. A factor of a graph G is any graph obtainable from G by suppressing edges but not vertices. An f-factor of G is a factor of G which is regular of degree f. Necessary and sufficient conditions for the existence of a 1-factor of G are known [W. T. Tutte, J. London Math. Soc. **22**, 107–111 (1947); these Rev. **9**, 297]. The author generalizes these to any f.

Let a "subdivision" of G be defined thus. First the vertices of G are arbitrarily divided into "s-points", "k-points" and "n-points". Then the n-points are divided into disjoint "P-sets" according to the rule that vertices in different P-sets are not joined by an edge. Let G' be the graph obtained from G by suppressing the s-points and the edges of which they are ends. Let the numbers of s-points, k-points and n-points be s, k and n respectively. Let Z be the sum of the degrees of the k-points. Write $F=kf-Z$. For each P-set let σ_i be the number of its elements and σ_e the number of edges of G' with one end in the P-set and the other not. The index of the P-set is defined as 1 or 0 according as $\sigma_e+f\sigma_i$ is odd or even. Let p be the sum of the indices of the P-sets.

The author proves that G has no f-factor if and only if there is a subdivision of G satisfying $F+p>sf$. He applies this theorem to regular graphs and obtains a sufficient condition, simpler in form, for a regular graph of degree g to have an f-factor. The author emphasizes the importance in his theory of the "hyper-f-prime" graphs. These generalize the "hyperprime" graphs used by the reviewer. They are graphs which have no f-factor but which acquire an f-factor whenever any two vertices (possibly identical) not already joined by f edges are joined by a new edge. The author first characterizes the hyper-f-prime graphs and then uses the simple result that any graph having no f-factor is a factor of some hyper-f-prime graph.
W. T. Tutte (Toronto, Ont.).

GEOMETRY

Durieu, M. Points remarquables du triangle. Mathesis **60**, 105–111 (1951).

Thébault, Victor. Bibliographie des triangles et des tétraèdres spéciaux. Bull. Soc. Roy. Sci. Liège **19**, 574–607 (1950).

Thébault, Victor. Sur des triangles associés. Mathesis **60**, supplement, 10 pp. (1951).

Thébault, V. Sur le tétraèdre dont les arêtes opposées sont deux à deux égales. Enseignement Math. **39** (1942–1950), 50–60 (1951).

Godeaux, Lucien. Remarque sur les tétraèdres de Moebius. Bull. Soc. Roy. Sci. Liège **20**, 14–15 (1951).

Kreul, H. *P*-Viereck, *S*-Viereck und Sehnenviereck. Math.-Phys. Semesterber. **2**, 139–150 (1951).

A *P*-quadrilateral is one in which the product of a pair of adjacent sides is equal to the product of the remaining two sides. It is shown that if two lines through two diagonally opposite vertices of such a quadrilateral meet on the other diagonal, their isogonal conjugates (the author does not use this accepted term) also meet on that diagonal, and conversely. The proposition is based on a known property of isogonal conjugates. An *S*-quadrilateral is one in which the angles which the diagonals form with the sides satisfy a certain relation. The author shows that the relation is satisfied by *P*-quadrilaterals, cyclic quadrilaterals, and no others.

N. A. Court (Norman, Okla.).

Deaux, R. Quartiques bicirculaires unicursales. Nieuw Tijdschr. Wiskunde **38**, 287–293 (1951).

Lorent, H. Sur des figures inscrites à une cubique de genre un ou à une biquadratique gauche de première espèce. Bull. Soc. Roy. Sci. Liège **19**, 425–428 (1950).

Lorent, H. Sur des relations entre groupes de points d'une cubique plane de genre un ou d'une biquadratique gauche de première espèce. Bull. Soc. Roy. Sci. Liège **19**, 417–424 (1950).

***Maxwell, E. A. General Homogeneous Coordinates in Space of Three Dimensions.** Cambridge, at the University Press, 1950. xiv+169 pp. $2.75.

This book is a sequel to the same author's "The Methods of Plane Projective Geometry Based on the Use of General Homogeneous Coordinates" [Cambridge University Press, 1946]. The ideas and methods used are to a great extent those of Baker [Principles of Geometry, especially vol. 3, Cambridge University Press, 1923] though frequent reference is also made to Todd [Projective and Analytical Geometry, Pitman, New York, 1946; these Rev. **9**, 524].

In chapter I points are defined by their homogeneous coordinates, and denoted by single symbols which are added, subtracted, and multiplied by scalars, in the manner familiar in vector algebra. These coordinates are throughout complex numbers, and no distinction is drawn between real and imaginaries, except in the plane Euclidean geometry in the appendix to this chapter. A line is defined as the aggregate of points satisfying syzygies with two given distinct points, and a plane as that of points satisfying syzygies with three given noncollinear points; the determination of a plane, line, or point by one, two, or three linear equations is derived from this. The homogeneous coordinates of a plane, the general linear transformation, and the principle of duality are simply treated, and a number of classical projective configurations (von Staudt's theorem, Desargues' theorem, the polar plane of a point with respect to a tetrahedron, Möbius tetrahedra, Desmic tetrahedra) are studied by means of syzygies between points. In an appendix the circle (or for $a=0$ the straight line)

$$a(x^2+y^2)+2px+2qy+r=0$$

of the Euclidean plane is represented by the point (a, p, q, r) of projective space, and it is shewn that the sides and altitudes of a general triangle, with 18 other circles related in a determinate way to the triangle, including the circumcircle, nine-points circle, etc., correspond to the vertices of two conjugate triads of Desmic tetrahedra.

The next two chapters deal with the quadric surface, defined in the first instance simply as the locus of points satisfying a quadratic equation. The polar theory, the quadric envelope of planes, and the various degenerate cases of both locus and envelope, are discussed. The generators are studied by means of the equation $xt=yz$, and its parametric solution $(\lambda\mu, \lambda, \mu, 1)$. The projective generation of the surface by two related ranges of points or pencils of planes, or by the lines meeting three lines, is examined.

Chapter IV deals with the six coordinates of a line, the linear complex and congruence, the generators of a quadric as lines common to three linear complexes, and the tetrahedral complex. No use is made of five-dimensional geometry (nor indeed anywhere in the book is anything mentioned outside of three dimensions). In chapter V the twisted cubic is treated by means of its parametric equations. Chapter VI deals with pencils and nets of quadrics (but not with invariants or the apolar relation). A few of the more important special cases are examined, but no general classification is attempted. In chapter VII elementary metrical properties are defined by taking an arbitrary plane to be at infinity and an arbitrary proper conic in this as absolute. After a few simple results about perpendicularity and spheres, most of the chapter is devoted to the study of confocal quadrics by dualising the properties of a general pencil. (Foci, however, are not treated.) Finally, chapter VIII gives an elementary account of matrix algebra, and illustrates its applications to quadrics, and also to line geometry, much in the style of Todd [op. cit.].

Throughout the book many simple corollaries are indicated without proof, under the name of theorem-examples. In addition there are impressive collections of examples at the end of each chapter, largely taken from Cambridge and London examinations, which are by no means trivial.

P. Du Val (Athens, Ga.).

Chong, F. Involutions on a conic and orthogonal matrices. J. Proc. Roy. Soc. New South Wales **83** (1949), 220–227 (1950).

Let s be a conic in a plane π. A point P of π not lying on s determines an involutory transformation **P** of s into itself. The author proves that a necessary and sufficient condition that the product **ABC**$\cdots$**PQ** be an involution is that Q lies on the cross-axis of the projectivity **ABC**$\cdots$**P**. By taking the equation of s in the form $x_1^2+x_2^2+x_3^2=0$ the author deduces various theorems concerning orthogonal 3×3 matrices and rotations in ordinary Euclidean space.

J. A. Todd (Cambridge, England).

***Kagan, V. F. Osnovaniya geometrii. Učenie ob obosnovanii geometrii v hode ego istoričeskogo razvitiya. Čast I. Geometriya Lobačevskogo i ee predistoriya. [Foundations of Geometry. A Study of the Foundations of Geometry in the Course of its Historical Development. Part I. The Geometry of Lobačevskiĭ and its Prehistory.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1949. 492 pp.

The author states that the appearance of this book is especially timely now that the teaching of the foundations of geometry has become a prescribed subject in the Russian universities and teachers colleges. Compared to other textbooks which have appeared in the last few years in the Russian and Ukrainian languages the book sets itself a wider task, having in view to help the student in a deeper

study of the foundations. The author, who published a two-volume work on the same subject more than forty years ago, has been teaching it for many years, and has contributed to its development. The book under review is the first volume of a new two-volume work whose second volume will be devoted in part to a systematic logical development of geometry on an axiomatic foundation. In this volume the historical and intuitive aspects of the subject are emphasized. Of the almost 500 pages the first 150 are devoted to the historical background. They contain the history of Greek geometry before Euclid with considerable space devoted to Aristotle's ideas on the logical structure of science. This is followed by a detailed critical exposition of Euclid's Elements with emphasis on the defects that gave rise to later developments. The rest of the first chapter is devoted to Greek mathematicians who continued Euclid's work and to a critical history of geometry textbooks, with the same emphasis. The second chapter deals with the "pre-history" of non-Euclidean geometry. Some of Legendre's proofs are given verbatim, and several false proofs of the parallel postulate are critically presented. The rest of the volume gives an exposition of non-Euclidean geometry following, in general, Lobačevskiĭ. It deviates from his methods only where later investigations have introduced essential simplifications, broader points of view, or extensions of his fundamental ideas. We note a chapter (13 pages) on existence proofs that do not use continuity (but use, for the most part, the axiom of Pasch), a chapter (39 pages) on lines of constant curvature, one (46 pages) on measurement of angles and areas, and another (48 pages) on analytic and differential geometry in the hyperbolic plane. In the chapter (29 pages) on geometrical constructions it is emphasized that it seems natural to allow in addition to ruler and compass the use of devices that permit the drawing of the other two lines of constant curvature, the equidistant and the oricircle, but it is brought out that their use does not lead to constructions impossible with ruler and compass only. Three-dimensional hyperbolic geometry, with special attention to equidistant surfaces and orispheres is presented in the last chapter (64 pages). The last seven pages deal with the investigations, more or less contemporary with those of Lobačevskiĭ, Gauss, Bolyai, Schweikert, and Taurinus.

G. Y. Rainich (Ann Arbor, Mich.).

van Gruting, C. J. Some remarks on properties of triangles and circles in elliptic geometry. I, II. Simon Stevin **27**, 153–176 (1950); **28**, 13–39 (1951). (Dutch)

Using Cayley's model of an elliptic metric in a projective plane, the author develops in part I the basic trigonometric relations for a triangle with respect to the elliptic metric. Also formulas for the area of a triangle and a circle are obtained. Part II contains the proofs of many theorems about triangles and circles, such as properties of medians, altitudes, bisectrices, inscribed and circumscribed circles, the circle of Hart of a triangle, Ptolemy's theorem, etc., all with respect to the elliptic metric. *W. van der Kulk.*

Haantjes, J. A characteristic local property of geodesics in certain metric spaces. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 66–73 (1951).

In a previous paper [same Proc. **50**, 496–508 = Indagationes Math. **9**, 302–314 (1947); these Rev. **9**, 53] the author defined a nonnegative number $\kappa(p)$ as the curvature of a rectifiable metric arc B at a point p if to each $\epsilon>0$ there corresponds a $\delta>0$ such that $|4![(l(q,r)-qr)/l^3(q,r)]-\kappa^2(p)|<\epsilon$ for every pair of points q, r of B with $pq<\delta$, $pr<\delta$, where $l(q,r)$ is the length of B between q, r and xy denotes the distance of points x, y. It was shown there that $\kappa(p)$ equals the Menger curvature $k(p)$ when the latter exists, but that the existence of $\kappa(p)$ does not imply that of $k(p)$. The present article gives a very simple proof of the important theorem that in a class of spaces which includes the Euclidean, elliptic, and hyperbolic, a rectifiable arc with everywhere vanishing curvature $\kappa(p)$ is a geodesic (that is, locally a segment). Subsumed under this theorem is the Menger-Schoenberg theorem that an arc B of a metric Ptolemaic space is a segment provided the Menger curvature vanishes at each point of B.

L. M. Blumenthal (Columbia, Mo.).

Ôtuki, Tominosuke. On paths in metric spaces. Math. Japonicae **2**, 9–22 (1950).

In a metric space the existence of oriented arcs with uniqueness properties of sufficiently small pieces of extremals in the calculus of variations is postulated. It is shown how on the basis of these axioms, curves in the large with the topological properties of extremals can be constructed. The nature of the sets beyond which prolongation starting from a given point, is, or is not, possible without losing uniqueness, is discussed. *H. Busemann.*

Convex Domains, Extremal Problems, Integral Geometry

***Aleksandrov, A. D. Vypuklye mnogogranniki. [Convex Polyhedra].** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 428 pp.

This work discusses problems of the following nature. What is the degree of freedom in prescribing certain data for a convex polyhedron, which then determine the polyhedron up to a certain elementary transformation (motion, translation, similitude)? The book is encyclopedic with regard to these questions, but does not treat problems like that solved by Steinitz' theory, because there uniqueness is a priori out of the question. The most novel feature, besides many new proofs and hitherto unpublished minor results, is the equal footing given to bounded and unbounded polyhedra. In both cases the number e of vertices, k of edges, and f of faces is assumed to be finite, but in the unbounded case some of the edges may be infinite rays, consequently some of the faces may extend to ∞. Doubly covered plane convex polygonal regions are considered as polyhedra to simplify the formulation of the theorems.

In the first chapter the definition and elementary properties of a convex polyhedron are treated. The plane scheme of a polyhedron is defined as a set of convex polygonal regions in the plane with given identification of edges and vertices. The elementary topology of convex polyhedra is developed; it is shown that a bounded polyhedron is homeomorphic to a sphere and that this is the case if and only if $e-k+f=2$. For an unbounded polyhedron homeomorphic to a plane $e-k+f=1$. This is mentioned here in order to indicate that practically no demands are made on the reader's knowledge. Chapter 2 describes the methods and results and proves, in two ways, Cauchy's lemma on the distribution of plus and minus signs. For a given plane scheme to be the scheme of a convex polyhedron it is obviously necessary that it have "positive curvature," that is, that the sum of the angles with the same vertex is at most

2π. It is shown in chapters 3 and 4 that for a given plane scheme with positive curvature and $e-k+f=2$, up to motions, exactly one bounded convex polyhedron belonging to this scheme exists. The problem is also solved for infinite polyhedra, but for uniqueness additional hypotheses become necessary. The number of actual edges may be smaller than k. The determination of the actual number of edges is carried through. Chapter 5 treats the analogous problems for convex polyhedra with a boundary.

Chapters 6, 7, and 8 treat Minkowski's theorem and related questions. Minkowski's theorem is given in the following more general form. If two bounded convex polyhedra P_1 and P_2 have parallel (oriented) faces, and if it is impossible to move by translation any face of P_1 (P_2) into a proper subset of the parallel face of P_2 (P_1), then P_2 may be obtained from P_1 by a translation. The proof is quite different from Minkowski's being based on Brouwer's theorem of the invariance of domain (proved in this book). The problem is also treated for unbounded polyhedra, but again additional hypotheses are necessary. Chapter 9 deals with polyhedra whose vertices lie on given rays issuing from one point. Chapters 10 and 11 treat infinitesimal rigidity. The problem was first formulated and solved by M. Dehn in 1915 with a method quite different from Cauchy's solution of rigidity in the large. It is shown here that Cauchy's method also allows one to solve the present problem. Similar questions are answered, as for instance the following. If a bounded polyhedron is so deformed that the intrinsic distances between the vertices remain stationary, then the deformation is initially (or infinitesimally) a motion.

The main part of the book is in large print and very easy to read since all proofs are carried out in detail. In addition, there are reports in small print on related fields like convex surfaces and extensions to spherical and hyperbolic space. These reports contain a wealth of material, but are frequently without proof and therefore definitely for the expert only. *H. Busemann* (Los Angeles, Calif.).

Leĭbin, A. S. On the deformability of convex surfaces with a boundary. Uspehi Matem. Nauk (N.S.) **5**, no. 5(39), 149–159 (1950). (Russian)

Let k' be the boundary of a convex set in E^3 with interior points (but not two parallel planes) and let k originate from k' by removing a finite number of disjoint closed sets, each bounded by a closed Jordan curve. Let C_k denote the convex closure of k and D_k the intersection of all half-spaces containing k and bounded by supporting planes of k at points of k (it is important to notice that the Jordan curves do not belong to k). With the methods of A. D. Aleksandrov the following result is established without any differentiability hypothesis: If C_k is a proper subset of D_k, then k is not rigid. This may also be formulated as follows: If a piece with positive integral curvature is removed from a convex surface, then the remainder is not rigid.

H. Busemann (Los Angeles, Calif.).

Prachar, K. Ein Satz über die Überdeckung des R_n mit Kugeln. Anz. Öster. Akad. Wiss. Math.-Nat. Kl. **1950**, 84–87 (1950).

Let Π be a parallelepiped in n-dimensional space, each of whose edges has length 1, and let A, B be any two different vertices of Π. The author proves that if P is any point of Π, then one at least of the distances PA, PB does not exceed $\{(n-1)^2+\frac{1}{4}\}^{\frac{1}{2}}$.

H. Davenport (London).

Sz.-Nagy, Gyula. Tschirnhaus'sche Eiflächen und Eikurven. Acta Math. Acad. Sci. Hungar. **1**, 36–45 (1950). (German. Russian summary)

Given n points $F_1, \cdots, F_n$ in the plane, respectively space. Each point F_k is provided with a positive weight q_k, $\sum q_k=1$. Let $r_k=r_k(P)$ denote the distance of the point P from F_k. The locus of the points P for which $R=\sum_1^n q_k r_k$ is constant is a convex Tschirnhaus curve, respectively surface, $E_n(R)$. There exists a number R_0 such that $E_n(R)$ is real if and only if $R \geqq R_0$ [cf. Loria, Spezielle algebraische und transzendente ebene Kurven, vol. I, 2d ed., Teubner, Leipzig, 1910, pp. 348–351]. The author gives simple proofs of known and new properties of the E_n's. It may be sufficient to quote the following, apparently new, result. Let O be the center of gravity of the points F_k, each of them provided with its weight q_k, and let $R>R_0$. Then the distance of each point of $E_n(R)$ from O is not greater than R. It reaches R only if the points F_k lie on a straight line. *P. Scherk.*

Aulbach, Helmut. Some geometrical inequalities for sets in Hilbert space. Proc. Amer. Math. Soc. **2**, 36–45 (1951).

The "side" $s^{[n]}(T_n)$ of an n-dimensional simplex T_n means the edge-length of a regular simplex of equal volume, and $s_n^*(S)=\sup s^{[n]}(T_n)$ for simplices T_n inscribed in the set S. For $n \geqq k$, $\Delta_{nk}(T_n)$ is the geometric mean of the "sides" of the k-dimensional faces of T_n, and $\Delta_{nk}(S)=\sup \Delta_{nk}(T_n)$ for T_n inscribed in S. Then $\Delta_{nk}(S)$ is monotonic decreasing in n, k. If $\Delta_k=\lim_{n\to\infty}\Delta_{nk}$ and $\sigma=\lim_{n\to\infty}\Delta_k$ then $s_k^*=\Delta_{kk}\to\sigma$. Δ_1 is the transfinite diameter of Fekete. For a sequence $\{\Sigma_n\}$ of spheres which cover S and whose radii $\rho_n\to 0$, the largest of the ρ_n is taken, say ρ_1, and the "transfinite radius" $\rho(S)$ of S is defined as $\inf \rho_1$ for all such coverings. The author has then two theorems: (I) for any bounded set S, $\sigma \geqq \rho\sqrt{2}$; (II) a necessary and sufficient set of conditions for S to be compact is that it be closed, bounded and of "transfinite side" $\sigma=0$. *H. D. Ursell* (Leeds).

Hlawka, Edmund. Ein Satz über additive Mengenfunktionen. Math. Nachr. **4**, 150–155 (1951).

Let $f(x)$ be the distance function of a symmetrical convex body K in m-dimensional space R_m. For $t \geqq 0$, let $h(t)$ be a continuous nondecreasing function with $h(0)=0$, $h(+\infty)>1$, and let $G(t)$ be a nonincreasing function such that $G(+0)=+\infty$, while $\int_0^h G(t)dh(t)$ is finite for all $h>0$. Let $P(E)$ be a positive countably additive set function defined for all measurable subsets E of a given measurable set M. The following generalization of a theorem of H. Cartan [see R. Nevanlinna, Eindeutige analytische Funktionen, Springer, Berlin, 1936, p. 136] is proved. There is a sequence of points x_i and numbers t_i with $\sum h(t_i) \leqq 5^m+1$ such that

$$\frac{1}{P(M)}\int_M G(f(x-a))dP < \int_{h=0}^{1} G(t)dh(t)$$

for all a in R_m which do not belong to the union of the sets $f(x-x_i) \leqq t_i$. If $P(E)$ corresponds to a mass distribution consisting of n point masses, the integrals $\int_0^h G(t)dh(t)$ and $\int_{h=0}^{1} G(t)dh(t)$ can be replaced by similar integrals having $h=1/n$ as the lower limit of integration.

The results are obtained by use of the following lemma in the special case when $t_1 \geqq t_2 \geqq \cdots$. Suppose that the system of convex sets K_i: $f(x-x_i) \leqq t_i$ $(i=1, 2, \cdots)$ is such that no set contains the centre of another set. Then it is possible to divide the system into 5^m subsystems such that no two sets of the same subsystem have a common point. The author

only considers the special case, but it is clear that the general case can be obtained from the special case by use of a diagonal process. *C. A. Rogers* (London).

Doss, Shafik. Inequalities characterizing the centre of mass of a material system. Proc. Math. Phys. Soc. Egypt **4**, no. 1, 53–59 (1949).

The center of mass a of a system S in 3-space is defined in the usual way, assuming that $\int_S |x_i| m(de) < \infty$, $i=1, 2, 3$. Let $m(S)=1$. If $|x-y|$ denotes the distance between points x and y in three-space, a is characterized uniquely by the inequality $|a-b|^p \leqq \int_S |x-b|^p m(de)$, with $p=1$, for all b. This suggests that if S is in an abstract metric space the center of mass should be defined as any point a with this minimizing property. Such a definition has been studied in a previous paper [C. R. Acad. Sci. Paris **226**, 1418–1419 (1948); these Rev. **9**, 520]. If S is a bounded set in 3-space a is also characterized uniquely by the same inequality with $p=2$. *J. L. Doob* (Urbana, Ill.).

Łojasiewicz, S. Sur la relation entre la largeur d'un contour plan et la déviation de ses arcs partiels. Ann. Soc. Polon. Math. **23**, 21–42 (1950).

The author sets out to establish, for a plane rectifiable contour of small width, and in particular for one which encloses a small area, the existence of a short arc on which the direction changes sharply. In so doing he obtains some much more general results, which lead him to an interesting conjecture.

In what follows, $\mathrm{Dev}(C)$ denotes, for any oriented simple arc C in the plane, the angle comprising the positive directions of all the chords of C, and $\mathrm{Wid}(J)$, the width of J, denotes, for any oriented simple closed plane curve J, the supremum of the diameters of circles contained in the domain interior to J. Further, for $0<\alpha<\pi$, $g(\alpha)$ denotes the infimum of $\mathrm{Wid}(J)$ for rectifiable J of length >1 such that $\mathrm{Dev}(C) \leqq \alpha$ for every subarc C of J of length 1. The author conjectures that $g(\alpha)=\cot(\frac{1}{2}\alpha)$ but proves this only with $\leqq$ in place of $=$. He succeeds, however, in showing that, for $0<\alpha<\pi$, (I) $g(\alpha)/\cot(\frac{1}{2}\alpha) \geqq 4^{-1} \sin\alpha[1+\cos(\frac{1}{2}\alpha)]^{-1}$ and in verifying further that $g(\alpha)$ decreases, that $g(\alpha)\to\infty$ as $\alpha\to 0$, and that $g(\alpha)>ng(n\alpha)$. The main theorems are substantially the geometrical restatements of these results. For instance (I) is equivalent to the statement: if J is a rectifiable simple closed plane curve of length $l \geqq d>0$ such that $\mathrm{Dev}(C) \leqq \alpha$ $(0<\alpha<\pi)$ for every arc C of J of length d, then J encloses a circle of diameter $\geqq \frac{1}{4}d\cos^2(\frac{1}{2}\alpha)[1+\cos(\frac{1}{2}\alpha)]^{-1}$. *L. C. Young* (Madison, Wis.).

Dinghas, Alexander. Isoperimetrische Ungleichungen für konvexe Polygone und Kurven mit Ecken in der Ebene und auf der Kugel. Math. Ann. **122**, 299–320 (1950).

Given a convex curve C of length L and of area F on the unit sphere. Let h denote the radius of the largest circle $\mathfrak{C}$ contained in C. By omitting from the domain bounded by C the points where distance from C is less than λ, we obtain the interior parallel domain bounded by the (spherically convex) curve $C(\lambda)$ $[0 \leqq \lambda < h]$. Let $L(\lambda)$ and $F(\lambda)$ denote its length and area respectively. The curve C has at most a countable number of vertices. Let $\pi-\alpha$ run through the angles formed by the one-sided tangents of C at these vertices and let $\pi-\alpha(\lambda)$ denote the corresponding angles of $C(\lambda)$. Put

$$F^*(h-\lambda)=2\int_\lambda^h \{\pi+\sum_a(\tan\tfrac{1}{2}\alpha(t)-\tfrac{1}{2}\alpha(t))\}\sin(h-t)\,dt.$$

Then (1) $F\cos h - L\sin h + F^*(h) \leqq 0$. This inequality is proved first in the case of a convex polygon C_n by showing that $F(\lambda)\cos(h-\lambda)-L(\lambda)\sin(h-\lambda)+F^*(h-\lambda)$ is then a monotonically increasing function of λ. Later C is approximated by C_n's. If $C=C_n$, then $F^*(h)$ is the area of a certain convex domain ["Kappenbereich"] which contains $\mathfrak{C}$. Hence $F^*(h) \geqq 2\pi(1-\cos h)$ for every C, and (1) implies Bonnesen's inequality

$$F\cdot(4\pi-F) \leqq L^2-(L\cosh-(2\pi-F)\sin h)^2.$$

The author determines those C's for which equality holds in (1). It holds for a C_n only if it is circumscribed to $\mathfrak{C}$. *P. Scherk* (Saskatoon, Sask.).

Hadwiger, H. Einige Anwendungen eines Funktionalsatzes für konvexe Körper in der räumlichen Integralgeometrie. Monatsh. Math. **54**, 345–353 (1950).

A function $\varphi(A)$ defined on all convex bodies A in E^3 is called additive if for a convex body $A\cup B$ formed by two convex bodies A and B whose intersection $A\cap B$ lies in a plane, the relation $\varphi(A)+\varphi(B)=\varphi(A\cup B)+\varphi(A\cap B)$ holds, and continuous if $A'\to A$ (in the usual sense of the theory of convex bodies) implies $\varphi(A')\to\varphi(A)$. If $\varphi(A)$ is, in addition, invariant under motions of A, then

$$\varphi(A)=\alpha\cdot 4\pi+\beta M(A)+\gamma F(A)+\delta V(A),$$

where $\alpha, \beta, \gamma, \delta$ depend on φ but not on A, and $M(A)$, $F(A)$, $V(A)$ are the integrals over the mean-curvature, the area, and the volume of A, respectively. The proof of this theorem will appear later. The present paper gives applications of the theorem. It is shown that various known formulas like Cauchy's formula for area and the main cinematic formulas of integral geometry, as well as some new results are easy consequences. The idea in applications is to choose or to find A for which $\alpha, \beta, \gamma, \delta$ can be readily evaluated. *H. Busemann* (Los Angeles, Calif.).

Kurita, Minoru. An extension of Poincaré formula in integral geometry. Nagoya Math. J. **2**, 55–61 (1951).

The author considers a homogeneous space with a Lie group and submanifolds which have an area in the sense of the reviewer [Ann. of Math. (2) **43**, 178–189 (1942); these Rev. **3**, 253]. For two such submanifolds of complementary dimensions, one fixed and the other moving, the integral over the kinematic density of the number of points of intersection is a constant multiple of the product of their areas provided that certain conditions are satisfied. The case of two line congruences in Euclidean 3-space is mentioned as an example for which these conditions are not satisfied. *S. Chern* (Chicago, Ill.).

Algebraic Geometry

***Welchman, W. Gordon. Introduction to Algebraic Geometry.** Cambridge, at the University Press, 1950. x+351 pp. $4.50.

This is a treatise on algebraic projective geometry of the plane with extensions to higher dimensions. Designed as an introduction to more advanced work, the author applies "to the more elementary problems the types of reasoning that are used in advanced work." The first three chapters deal with fundamental concepts and methods, including dimension, homogeneous coordinates, projective algebraic

systems, cross ratio, algebraic correspondences and the theory of rational algebraic systems. In chapters four to seven, inclusive, conics are studied chiefly by methods applicable to configurations of higher order and dimension. In chapters eight and nine, the projective properties of pairs of conics are discussed and in the next chapter the properties of (2, 2) correspondences are shown to be associated with those of pairs of conics. In chapter eleven, matrix algebra is applied to the study of conics, and in the last chapter, invariants and covariants of a conic and of sets of conics are treated.

The book would be better adapted to its purpose if it contained problems for the student to solve. A student should learn to "read" a mathematical treatise with pencil and paper at hand, working out everything in greater detail than usually can be given in the text. In a beginning course, especially, problems to be solved involving extensions and applications of the theory are essential to start the student on this method of study. The book is well written, the expositions clear and ample. The printing is excellent and the formulas well displayed. The methods are almost entirely algebraic except for a few synthetic proofs, chiefly in chapter five. Illustrative drawings are given where needed. A detailed table of contents partially makes up for the lack of an index. As a whole, the book is well suited to achieve a worthwhile goal, to "help some students to appreciate the volumes on Principles of Geometry [6 volumes, Cambridge University Press, 1922–1933] by Professor H. F. Baker."

T. R. Hollcroft (Aurora, N. Y.).

Lorent, H. Sur les intersections d'une cubique plane de genre un avec une autre courbe. Bull. Soc. Roy. Sci. Liège 20, 60–66 (1951).

Edge, W. L. A plane quartic with eight undulations. Proc. Edinburgh Math. Soc. (2) 8, 147–162 (1950).

Two types of plane quartic with 12 undulations are known, the first being Dyck's quartic $x^4-y^4-z^4=0$, and the second that discovered recently by Edge [Edinburgh Math. Notes no. 35, 10–13 (1945), p. 11; these Rev. 7, 324]. Masoni, in his original paper on quartics with undulations [Rend. Accad. Sci. Fis. Mat. Napoli (1) 21, 45–69 (1882)], missed the second type because of a surprising flaw in his reasoning. The present paper explains this and throws much new light on the whole question by showing how to construct geometrically quartics with only 8 undulations, and by proving that these can be specialized into quartics with 12 undulations in two distinct ways. The construction derives a unique quartic with 8 undulations from a conic and a line-pair in the plane, and the author obtains the equation of a family of such curves in the form

$$x^4-y^4-z^4+4fx^2yz+2f^2y^2z^2=0.$$

In this family, excluding the degenerate curves given by $f^4=1$, the only values of f which give curves with more than 8 undulations are (i) $f=0$, which gives Dyck's quartic, and (ii) those given by $f^4=81$, which give quartics of Edge's type, also possessing 12 undulations. The geometry of the family is exhibited in detail.

J. G. Semple (London.).

Adam, Denise. Sur deux surfaces du quatrième ordre. Bull. Soc. Roy. Sci. Liège 20, 39–47 (1951).

Manara, Carlo Felice. Sulla esistenza di curve algebriche piane irriducibili aventi dati caratteri plückeriani. Boll. Un. Mat. Ital. (3) 6, 9–14 (1951).

It is known that a plane curve of order 8 with 16 cusps cannot exist [O. Zariski, Amer. J. Math. 53, 309–318 (1931)]. In the present paper an example of a plane curve of order 8 and genus 5 with 14 cusps and 2 nodes is constructed. This contradicts the assertion of R. Apéry [C. R. Acad. Sci. Paris 214, 340–341 (1942); these Rev. 4, 167] to the effect that such a curve cannot have more than 13 cusps. The construction is as follows. If F_4 is a surface of 4th order with a nodal line r, and outside of r two cuspidal and two ordinary double points, then the first polar of F_4 with respect to a generic point O of F_4 meets F_4 outside of r in a curve φ_{10} with a double point at O. Projection of φ_{10} from O onto a general plane yields the required curve.

H. T. Muhly (Iowa City, Iowa).

Šilov, G. E. Singular points of algebraic curves in the plane. Uspehi Matem. Nauk (N.S.) 5, no. 5(39), 180–192 (1950). (Russian)

In dieser Arbeit werden die Gedankengänge, die sich mit dem Newtonschen Polygon und der bekannten Puiseuxschen Reihenentwicklung algebraischer Funktionen beschäftigen, wieder lebendig gemacht. Dabei setzt der Verfasser einerseits die betrachtete Kurve etwas allgemeiner als algebraisch voraus, nämlich in der Gestalt $f=\sum a_{k_i l_j}x^{k_i}y^{l_j}=0$, wobei nur die k_i als ganz, die l_j nur als reell vorausgesetzt werden; anderseits interessiert er sich nur für reelle Zweige dieser Kurve. Bei der Herleitung der Reihenentwicklungen in einem Punkt wird das Puiseuxverfahren in folgender Weise gewissermassen dualisiert: Statt dem einzelnen Gliede in $f=0$ den Punkt (k_i, l_j) des Diagramms zuzuordnen, wird ihm die Gerade $n=k_i+rl_j$ (n und r kartesische Koordinaten) zugeordnet. Aus Stücken dieser Geraden setzt man dann einen geschlossenen Streckenzug zusammen, dessen Eckpunkte zur Bestimmung des ersten Exponenten r der Reihenentwicklung $y=Ax^r(1+y_1)$ dienen. Zur Bestimmung von A ergibt sich dann wie bei Puiseux eine algebraische Gleichung, auf deren reelle Wurzeln es hier aber nur ankommt. Zur Berechnung der weiteren Exponenten und Koeffizienten ist das Verfahren dann wie üblich, zu iterieren. Erläutert wird alles an den Kurven: $x^4+x^3y-y^3-xy=0$, $xy^5+y^6-2y^5+y^4+xy^3-2x^3y^2+x^5y-x^6=0$. *W. Burau.*

Masotti Biggiogero, Giuseppina. Sulle singolarità della curva hessiana. Ann. Mat. Pura Appl. (4) 30, 277–289 (1949).

Dans ses travaux antérieurs [Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 10(79), 89–96 (1946); ces Rev. 10, 398] l'auteur a étudié la hessienne d'une courbe ayant un point double de rebroussement ou tacnodal d'espèce quelconque, mais générique en ce sens que la branche superlinéaire ou les deux branches linéaires qui en sont issues n'ont pas d'inflexion. Dans le mémoire actuel, l'auteur étudie le cas d'un point d'ordre plus élevé, générique, dans le même sens du mot. Une courbe ayant un point r-uple avec r tangentes distinctes (sans inflexion) donne pour la hessienne $3r-4$ branches linéaires distinctes; sans toutefois que les tangentes soient nécessairement toutes distinctes comme l'avait cru C. Segre [Giorn. Mat. Battaglini 36, 1–50 (1898)]: un contre-exemple est donné.

Lorsqu'une courbe admet un point d'ordre r à tangentes toutes confondues suivant t, mais avec contact $(r+1)$-ponctuel, un résultat de Brill [Math. Ann. 13, 175–182

(1878)] assure que sa hessienne admet ce point avec l'ordre $3r-3$ et $2r-2$ tangentes confondues suivant t. L'auteur précise la structure de cette singularité, qui est la superposition de deux branches superlinéaires d'ordres respectifs r et $r-2$, de classe 1, tangentes à t, et de $r-1$ branches linéaires distinctes, non tangentes à t. Lorsqu'une courbe C admet un point d'ordre r à branches linéaires, sans inflexion, ayant deux à deux un contact d'ordre s (et non supérieur), sa hessienne admet en ce point $3r-3$ branches linéaires distinctes ayant entre elles, et avec celles de C, un contact d'ordre s (et non supérieur). Ce dernier résultat complète et précise celui obtenu par l'auteur dans une étude antérieure [Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 7(76), 271–280 (1943); ces Rev. 8, 343]. *L. Gauthier* (Nancy).

Longhi, Ambrogio. Sulle curve razionali dotate di cuspidi. Ann. Mat. Pura Appl. (4) 29, 171–175 (1949).

On trouve ici un résumé de quelques résultats de géométries numérative sur les courbes rationnelles d'un hyperespace, que l'auteur démontre dans deux mémoires qui vont être publiés [pour le premier voir l'oeuvre analysé ci-dessous]. Il s'agit surtout d'une formule de De Jonquières se rapportant à une série algébrique γ_n^r, de dimension r et d'index ν, de groupes de n points sur une courbe rationnelle Γ; formule donnant le nombre N des groupes de la γ_n^r qui contiennent t points multiples selon ν_i+1 $(1\leqq t\leqq n-r; i=1, 2, \cdots, t)$, où $\nu_1+\nu_2+\cdots+\nu_t=r$. Cette formule ne s'applique pas lorsque Γ possède des points de rebroussement, car alors le nombre ρ de ces points figure dans N d'une manière inconnue. L'auteur donne une nouvelle formule qui fait connaître la valeur de N en y comprenant seulement les groupes de γ_n^r dont les points multiples sont tous différents des points de rebroussement de Γ. On peut faire différentes applications de cette nouvelle formule; les plus intéressantes s'obtiennent en l'appliquant à la série linéaire des hyperplans osculateurs d'une courbe Γ (d'ordre n dans un espace S_r) qui passent par les différents points d'un espace S_ρ de S_r. Par cette voie, on peut trouver, par exemple, les ordres de toutes les variétés multiples des différentes développables osculatrices d'une courbe rationnelle Γ, et les ordres des variétés d'intersection de ces mêmes variétés, soit dans le cas d'une courbe Γ générale, soit dans le cas où Γ possède ρ points de rebroussement; pour $\rho=0$ on retrouve des résultats connus. *E. G. Togliatti.*

Longhi, Ambrogio. I gruppi con elementi multipli distinti dalle cuspidi nelle serie algebriche sulle curve razionali cuspidate. Comment. Math. Helv. 24, 196–203 (1950).

Si l'on considère sur une courbe algébrique C de genre p une série algébrique γ_n^r, d'ordre n, de dimension r, et d'index $\nu\geqq 1$, composée de groupes de points équivalents, on peut demander quel est le nombre des groupes de la série qui possèdent des points multiples avec des multiplicités données. La réponse à cette question est donnée par une formule de De Jonquières, à laquelle Severi et Torelli ont apporté des compléments. L'application de cette formule n'est pas toujours facile; car, si la courbe C possède des points de rebroussement, on ne sait pas combien des groupes cherchés sont compris parmi ceux qui contiennent un point de rebroussement. Pour vaincre cette difficulté, l'auteur donne ici, dans le cas $p=0$, une formule générale qui donne le nombre des groupes de la série γ_n^r possédant chacun des points multiples (avec des multiplicités données) tous différents des points de rebroussement. On suppose de connaitre le nombre ρ de ces derniers points. L'énoncé du résultat est rapporté à une variété algébrique générale C, simplement infinie, irréductible et rationnelle. *E. G. Togliatti.*

Piazzolla-Beloch, Margherita. Teoria diametrale delle curve algebriche piane. Ann. Univ. Ferrara. Parte I. 7, 39–121 (1948).

This is a reprint with different pagination of a paper with the same title [Instituto di Matematica dell'Università di Ferrara, 1949; these Rev. 11, 684]. *P. Du Val.*

Rosina, Bellino Antonio. Sulle curve algebriche piane con punto principale indeterminato. Ann. Univ. Ferrara. Parte I. 7, 195–212 (1948).

A diameter of a plane curve being defined as the locus of mean centres of the sets of points traced by a system of parallel lines, i.e. the polar line of a point at infinity, and a principal point as one (if there is one) through which all the diameters pass, the author considers the case in which there are an infinity of principal points, and shews that in this case all the diameters coincide, and are the locus of principal points. The algebraic condition for this is that the terms of the two highest degrees (in nonhomogeneous coordinates) are the nth and $(n-1)$th powers of the same linear expression. The curve has a double point at infinity in the diametral direction, and all intersections with the line at infinity coincide. Cubics and quartics satisfying this condition are examined in detail; the former are Wallis cubics, with a cusp at infinity, the line at infinity being the tangent there; the direction of the y axis being that to the cusp, the equation of the curve is of the form: $y=$cubic in x; there is one inflexion, which is a centre of symmetry, and the unique diameter joins this point to the cusp. In the case of the quartic there is a tacnode at infinity, again with the line at infinity as tangent. The equation is of the form:

$$(\text{linear in } x, y)^2=\text{quartic in } x,$$

and the linear expression on the left equated to zero gives an axis of "oblique symmetry." There are 19 sketches of different forms of the curve, chiefly distinguished by different relations between the roots of the quartic on the right, and including also some cases in which x is absent on the left, so that the oblique symmetry becomes orthogonal.

P. Du Val (Athens, Ga.).

Roselli, Alberto. Sulla topologia delle curve situate su un cono cubico. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 9, 233–238 (1950).

On the odd sheet of a real cubic cone of vertex V there can be three types of nonsingular even circuit: 1. Even on the generators (i.e. meeting each generator in an even number of points), not passing through V, and continuously deformable into a small circuit round a general point. 2. Even on the generators, not passing through V, and continuously deformable into a small circuit round V. 3. Odd on the generators, passing through V. [The author states correctly that a circuit of type 2 must meet all the generators, but incorrectly that one of type 1 does not meet them all; what is true is that a circuit of type 1 cannot meet them all in the same number of points, i.e., must touch some of them, whereas one of type 2 may perfectly well meet all generators in the same number of points and touch none.] Odd circuits are of two kinds, odd on the generators, not passing through V, and even on the generators, passing through V. Two even circuits, or one even and one odd, have an even number of intersections, two odd circuits an odd number. Hence if the cone has only its even sheet, any

nonsingular curve of even (odd) order lying on it has no (precisely one) odd circuit, respectively, all (other) circuits being even. *P. Du Val* (Athens, Ga.).

Roselli, Alberto. Sulla topologia delle curve situate su un cono cubico. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **9**, 324–326 (1950).

Continuing from paper I [see the preceding review], if the cone has an even sheet, every odd circuit and no even circuit on this passes through V, and either may be even or odd on the generators. Two even circuits have an even number of intersections, an even and an odd circuit have an even or odd number according as the odd circuit is even or odd on the generators, and two odd circuits an even or odd number (apart from V) according as they are of like or unlike parity on the generators. Thus on the cone with two sheets (one odd and one even), a nonsingular curve of even order may have either no odd circuit or precisely two, one on each sheet; a nonsingular curve of odd order has precisely one odd circuit, which may be on either sheet.

P. Du Val (Athens, Ga.).

Godeaux, Lucien. Sur la construction de certaines quartiques rationnelles. Mathesis **60**, 81–84 (1951).

Godeaux, Lucien. Sur les courbes-base d'un système linéaire de surfaces. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **37**, 202–206 (1951).

Godeaux, Lucien. Sur les courbes tracées sur une surface multiple. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 678–682 (1950).

Soit F une surface algébrique contenant une involution cyclique I d'ordre premier impair p, et n'ayant qu'un nombre fini de points unis; un de ces points unis, de 2^e espèce, soit A. Sur une surface ϕ, multiple d'ordre p, image de I, au point A correspond un point de diramation isolé A'; ce point A' équivaut à un ensemble de courbes rationnelles

$$(1) \qquad \sigma_1\rho_1'\cdots\rho_r'\tau_1\rho_1\cdots\rho_t\tau_2\rho_1''\cdots\rho_s''\sigma_2$$

dont chacune rencontre la précédente et la suivante en un point, mais ne rencontre pas les autres. Les courbes $\rho_i, \rho_i', \rho_i''$ ont le degré virtuel -2. En utilisant certaines propriétés des courbes tracées sur ϕ, l'auteur démontre une relation qui lie entre eux les nombres r, s, t, l'ordre p de I et les degrés virtuels $-\xi_1, -\eta_1, -\eta_2, -\xi_2$ de $\sigma_1\tau_1\tau_2\sigma_2$. Cette relation vaut dans le cas où le cône tangent à ϕ en A' se décompose en quatre parties; la relation même se simplifie lorsque ces parties se réduisent à trois. *E. G. Togliatti* (Gênes).

Godeaux, Lucien. Structure des points de diramation d'une surface multiple d'ordre 29. Acad. Roy. Belgique. Cl. Sci. Mem. Coll. in 8°. (2) **24**, no. 7(1597), 36 pp. (1950).

As an example of the application of his general results on the isolated fixed points of a cyclic transformation on an algebraic surface F, and the corresponding isolated branch points of a surface Φ, image of the cyclic involution on F generated by the transformation [Ann. Sci. École Norm. Sup. (3) **65**, 189–210 (1948); Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 15–30, 270–284, 285–292, 532–541, 636–641, 828–833, 834–840 (1949); these Rev. **11**, 392; **10**, 735; **11**, 205, 392, 393, 538; and the paper reviewed below] the author considers the case of an involution of order 29. A fixed point A on F is of the first or second kind according as every point in its neighborhood is fixed, or the neighborhood is transformed into itself by a homography:

$$\lambda' : \mu' = \epsilon\lambda : \epsilon^\alpha\mu = \eta^\beta\lambda : \eta\mu$$

where $\epsilon, \eta = \epsilon^\alpha$ are primitive 29th roots of unity and α, β are reciprocal residues (mod 29). If A is of the first kind, the corresponding point A' on Φ is 29-ple with rational tangent cone. If A is of the second kind, A' is of lower multiplicity (from 2 to 15 in different cases) the tangent cone being reducible, in one case to as many as four constituents, these partial tangent cones being all rational, and arranged in a natural sequence such that consecutive cones have a common generator; on a surface free from singularities A' and its neighborhoods are mapped by a sequence of rational curves of grade not greater than -2, consecutive curves having a point in common. Some of these correspond to constituents of the tangent cone (namely the two end curves of the sequence and all that have grade less than -2), the rest (if any) to singular points (conic nodes or binodes with or without further singularities in their neighborhoods) consecutive to A' on one or more simple branches, each touching a common generator of two constituents of the tangent cone. The fourteen cases corresponding to the different pairs of reciprocals (α, β) are studied in detail.

P. Du Val (Athens, Ga.).

Godeaux, Lucien. Sur certaines surfaces multiples n'ayant qu'un nombre fini de points de diramation. Ann. Mat. Pura Appl. (4) **28**, 89–106 (1949).

Après avoir rappelé la méthode par laquelle il a pû déterminer la structure des points unis non parfaits d'une involution cyclique I_p sur une surface algébrique S et la structure de la singularité au point de diramation correspondant, de la surface image de I_p [Ann. Sci. École Norm. Sup. (3) **65**, 189–210 (1948); ces Rev. **11**, 392] l'auteur l'applique aux deux cas suivants, où I_p ne possède qu'un nombre fini de points unis: en un tel point, les tangentes à S subissent une homographie

$$t' = t \exp 2\frac{k}{p}i\pi \qquad (k \neq 0,\ p \text{ premier}).$$

Lorsque $p = 6\eta+1$ et $k = 3\eta+1$, le point de diramation est d'ordre $\eta+2$ et le cône des tangentes en ce point est constitué d'une cône rationnel d'ordre $\eta-1$, d'un cône du second ordre qui le rencontre suivant une génératrice, et d'un plan qui rencontre le premier cône suivant une génératrice, sans rencontrer le second. Un exemple est donné de surface S contenant une involution I_p dont tous les points unis sont de cette nature. Lorsque $p = 6\eta-1$ et $k = 3\eta$, le point de diramation est d'ordre $\eta+1$ et le cône des tangentes en ce point est constitué d'un cône d'ordre $\eta-1$ et de deux plans qui ne se rencontrent pas mais rencontrent chacun le cône suivant une génératrice: Sur l'une d'elles la surface image admet en outre un point double conique infiniment voisin du point de diramation. Un exemple est également donné de surface S contenant une involution I_p dont tous les points unis sont de cette nature. *L. Gauthier* (Nancy).

Godeaux, Lucien. Détermination des singularités d'une surface multiple en certains points de diramation. Ann. Sci. École Norm. Sup. (3) **67**, 1–13 (1950).

Comme application de la méthode qu'il a mise au point pour étudier la structure des points unis non parfaits d'une involution cyclique I_p d'une surface algébrique S [mêmes Ann. (3) **65**, 189–210 (1948); ces Rev. **11**, 392; cf. aussi l'analyse ci-dessus], l'auteur étudie trois exemples con-

duisant, sur la surface image de I_p, à des points de diramation présentant des singularités assez complexes. Dans les trois exemples on obtient un point de diramation triple triplanaire; deux des plans tangents T_1, T_2 sont incidents au troisième T_3 sans l'être entre eux. Dans le premier exemple, qui correspond à $p=17$, la surface image admet en outre sur la droite T_1T_3 un point double biplanaire infiniment voisin du point triple. Dans le second exemple, qui correspond à $p=23$, la surface image admet un point double biplanaire sur T_1T_3 et un point double conique sur T_2T_3 infiniment voisins du point triple. Dans le troisième exemple où l'on a encore $p=23$, la surface image admet sur T_1T_3 deux points doubles biplanaires infiniment voisins successifs au point triple. *L. Gauthier* (Nancy).

Godeaux, Lucien. Étude de certains points de diramation isolés de surfaces multiples. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 368–382 (1950).

L'auteur introduit un point de vue nouveau dans l'étude de la structure des points unis non parfaits d'une involution cyclique I_p sur une surface algébrique S et celle de la structure de la singularité au point de diramation correspondant, de la surface Φ image de I_p. [Voir les deux oeuvres analysés ci-dessus et l'oeuvre cité].

Soit, dans un modèle projectif de S où I_p est déterminée par une homographie cyclique de l'espace ambiant, A un point uni non parfait. Dans le mémoire cité dans l'analyse ci-dessus l'auteur a utilisé des systèmes $|C^i|$ de courbes appartenant à I_p, découpées sur S par les hyperplans passant par les axes ponctuels $\sigma_1 \cdots \sigma_{p-1}$ de l'homographie génératrice, non sécants à S, et présentant pour i croissant, des multiplicités croissantes en A, caractéristiques de sa structure. Au point de diramation A' correspondant à A, est ainsi associée sur Φ la suite des systèmes $|\Gamma^i|$ images des $|C^i|$.

Dans le mémoire actuel, l'auteur, remarquant que $|\Gamma^1|$ est le système des sections de Φ par les hyperplans passant par A', introduit les systèmes $|\overline{\Gamma}^i|$ des sections de Φ par les hyperplans qui contiennent à la fois A' et ses $i-1$ premiers points doubles infiniment voisins successifs. Le présent mémoire comporte l'étude détaillée et complète d'un exemple, dans lequel $p=31$, où les deux suites $|\Gamma^i|$ et $|\overline{\Gamma}^i|$ sont distinctes: Φ, d'ordre n, admet A multiple d'ordre 7 avec 4 points doubles infiniment voisins successifs, et les systèmes $|\Gamma^i|$ ont pour degrés respectifs $n-7$, $n-9$, $n-10$, $n-13$. *L. Gauthier* (Nancy).

Godeaux, Lucien. Sur un point de diramation heptuple d'une surface multiple. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 672–677 (1950).

Cette note se rapporte à une recherche du même auteur sur les points de diramation isolés d'une surface algébrique multiple, dont il a déjà été question plusieurs fois [par ex., même Bull. (5) **35**, 828–833, 834–840 (1949); ces Rev. **11**, 538]. Il s'agit ici simplement d'un exemple d'un point de diramation isolé A' d'une surface multiple ϕ qui représente une involution I d'ordre 83 sur une surface algébrique F; le point A' vient d'un point uni isolé A de I. Dans le domaine du point A on a sur F l'homographie $x_0' : x_1' : x_2' = x_0 : \epsilon x_1 : \epsilon^{66} x_2$, où ϵ est une racine primitive d'ordre 83 de l'unité. Le cône tangent à ϕ en A' est d'ordre 7 et se décompose en quatre parties $\sigma_1, \tau_1, \tau_2, \sigma_2$; dont la première est un cône rationnel d'ordre 4 tandis que les autres parties sont trois plans. Sur la droite commune aux plans τ_2, σ_2 il y a dans ce cas un point double conique infiniment voisin de A'. *E. G. Togliatti.*

Godeaux, Lucien. Remarque sur la jacobienne d'un système de surfaces. Bull. Soc. Roy. Sci. Liège **19**, 379–381 (1950).

Godeaux, Lucien. Sur les homographies cycliques dont la période est une puissance de deux. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 853–858 (1950).

Let H be a cyclic collineation of period $2m$ in S_{2m-1}, with isolated united points, where $m=2^{n-1}$. By considering the intersection of a set of suitably chosen quadrics, each invariant under H, the author constructs a surface F, of order 2^{2n-3}, which carries a cyclic involution of order 2^n free from united points. The image surface Φ of this involution has divisor $\sigma=2^n$. *J. A. Todd* (Cambridge, England).

Godeaux, Lucien. Étude de quelques surfaces algébriques. Bull. Soc. Roy. Sci. Liège **19**, 448–468 (1950).

Lense, Josef. Über die Hessesche Fläche einer allgemeinen Fläche dritter Ordnung. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1949**, 13–18 (1950).

An irreducible quartic surface with nodes at the 10 vertices of a proper pentahedron is the Hessian of a unique cubic surface. *J. G. Semple* (London).

Goddard, L. S. On a class of determinantal primals and their multiple loci. Proc. Cambridge Philos. Soc. **47**, 286–298 (1951).

As the characteristic equation $|\lambda E - X| = 0$ of a square matrix X of order n (E being the unit matrix) is homogeneous in λ and the elements of X, it can be regarded as the equation of a primal (hypersurface) of order n in space Σ of n^2 dimensions, on which are a sequence of singular loci of increasing multiplicity given by equations $|\lambda E - X|_r = 0$ ($r=0, \cdots, n-1$), i.e., the vanishing of all minors of order $r+1$ in the determinant. If X is subject to a condition of the type $GX + X'G = 0$, where G is a given matrix and X' is the transpose of X, this gives linear relations between the elements of X, and thus a linear section of the primal. The values of G considered are as follows.

(i) $G=0$, X is a general matrix M; $D^{(r)}$ is the locus given by $|\lambda E + M|_r = 0$, $K^{(r)}$ is that given by $|M|_r = 0$, in the prime $\tilde{\omega}$ ($\lambda = 0$); $D^{(0)}$ is the point $(-1, E)$, and $D^{(r)}$ the cone projecting $K^{(r)}$ from $D^{(0)}$. (The loci $D^{(r)}$ are studied at large by Room [The Geometry of Determinantal Loci, Cambridge University Press, 1938] to which work frequent reference is made.)

(ii) $G=E$, X is a general antisymmetric matrix S, so that the loci are in $\frac{1}{2}n(n-1)$ dimensions. $\mathfrak{P}^{(n-r)}$ is the locus given by $|\lambda E + S|_{n-r} = 0$, $\Omega^{(n-r)}$ that given by $|S|_{n-r} = 0$ in $\tilde{\omega}$, and $\overline{\mathfrak{P}}^{(n-r)}$ the component of $\mathfrak{P}^{(n-r)}$ which does not lie wholly in $\tilde{\omega}$. For n even, $\mathfrak{P} = \mathfrak{P}^{(n-1)}$ is irreducible, $\Omega^{(n-1)}$ being its section by $\tilde{\omega}$; $\overline{\mathfrak{P}}^{(n-2)}, \cdots, \overline{\mathfrak{P}}^{(\frac{1}{2}n+1)}$ are irreducible but of lower dimensions than the corresponding $\Omega^{(n-r)}$; $\mathfrak{P}^{(\frac{1}{2}n)}$ consists of two identical irreducible components, and $\overline{\mathfrak{P}}^{(\frac{1}{2}n-1)}, \cdots, \overline{\mathfrak{P}}^{(1)}$ are null. For n odd, $\mathfrak{P}$ breaks up into $\tilde{\omega}$ and an irreducible primal $\overline{\mathfrak{P}}$ of order $n-1$, $\overline{\mathfrak{P}}^{(n-2)}, \cdots, \overline{\mathfrak{P}}^{(\frac{1}{2}(n+1))}$ are irreducible, and $\overline{\mathfrak{P}}^{(\frac{1}{2}(n-1))}, \cdots, \overline{\mathfrak{P}}^{(1)}$ are null. In either case $\overline{\mathfrak{P}}^{(n-r)}$ (where it exists) is generated by $\infty^{\frac{1}{2}r(2n-2r-1)}$ spaces of $\frac{1}{2}n(n-2r-1)+\frac{1}{2}r(r+1)$ dimensions, and is thus itself of $\frac{1}{2}n(n-1)-r^2$ dimensions.

(iii) (for even values of n only) $G=I$, a matrix with $\frac{1}{2}n$ blocks of the form $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ down the diagonal and zeros elsewhere. Then X is of the form $I\mathbb{S}$, where $\mathbb{S}$ is symmetric, so that the loci are in $\frac{1}{2}n(n+1)$ dimensions. $\mathfrak{P}^{*(n-r)}$ is the locus given by $|\lambda E - I\mathbb{S}|_{n-r} = 0$, $y^{(n-r)}$ that given by $|\mathbb{S}|_{n-r} = 0$ in

$\tilde{\omega}$, and $\overline{\mathfrak{P}}^{*(n-r)}$ the component of $\mathfrak{P}^{*(n-r)}$ which is not wholly in $\tilde{\omega}$. Then $\mathfrak{P}^*=\mathfrak{P}^{*(n-1)}$ is irreducible, $y^{(n-1)}$ being its section by $\tilde{\omega}$, $\overline{\mathfrak{P}}^{*(n-2)}, \cdots, \overline{\mathfrak{P}}^{*(1^n)}$ are irreducible, and generated by $\infty^{\frac{1}{2}r(2n-3r+1)}$ spaces of $\frac{1}{2}(n-r)(n-r+1)$ dimensions, and are thus of $\frac{1}{2}n(n+1)-r^2$ dimensions, which is less than that of the corresponding $y^{(n-r)}$; $\overline{\mathfrak{P}}^{*(1^{n-1})}, \cdots, \overline{\mathfrak{P}}^{*(1)}$ are null.

(iv) Finally, for even values of n the intersection $\mathcal{P}$ of $\mathfrak{P}$ with $\mathfrak{P}^*$ is considered, i.e. the primal given by $|\lambda E-I\mathbb{S}|=0$, where $\mathbb{S}$ is symmetric and $I\mathbb{S}$ antisymmetric. This means that $\lambda E-\mathbb{S}$ is obtained from a matrix of the form

$$\begin{pmatrix} \lambda E-S & T \\ -T & \lambda E-S \end{pmatrix},$$

where S, T are respectively antisymmetric and symmetric matrices of order $\frac{1}{2}n$, by writing the rows of the upper half and those of the lower half alternately, and making the same permutation of the columns; thus $\mathcal{P}$ is in $\frac{1}{4}n^2$ dimensions. $\mathcal{P}$ breaks up into two cones $\mathcal{P}_1$, $\mathcal{P}_2$, given by $|\lambda E-S\pm iT|=0$, whose vertices are the points $\lambda=\pm i$, $S=0$, $T=E$. [In theorem 3 "two-point cones" is a misprint for "two point-cones."] Their intersection consists of K^*, the section of both by $\tilde{\omega}$, and a residual locus of order $\frac{1}{2}n(\frac{1}{2}n-1)$ and $\frac{1}{4}n^2-2$ dimensions which is the complete intersection of two primals; each has on it a double locus which is a cone of order $\frac{1}{12}(\frac{1}{4}n^2)(\frac{1}{4}n^2-1)$ and $\frac{1}{4}n^2-4$ dimensions. [In the text, the notations n, $2n$, $2r$ are used rather confusingly for the order of the matrices. In this review n is adhered to throughout.] *P. Du Val* (Athens, Ga.).

Jongmans, F. Extension d'une borne inférieure pour le genre linéaire des surfaces algébriques. Bull. Soc. Roy. Sci. Liège **19**, 250–266 (1950).

It is shewn that every algebraic surface satisfies the inequality $p^{(1)}\geqq 4p_g-p_a-6$ (where p_g, p_a, $p^{(1)}$ are as usual the geometric, arithmetic, and linear genera) with the following exceptions, of which nos. 1, 2, 4, 6 are known to exist, and the others (marked with *) are hypothetical. (1) Ruled surfaces of genus $\geqq 2$. (2) Surfaces on which the variable part of the canonical system is compounded with an irrational pencil of elliptic curves. (*3) Surfaces on which the variable part of the canonical system is compounded with an irrational pencil of curves of genus 2, the genus of the pencil being $p_g-p_a-\epsilon$ ($\epsilon=0, 1, 2$), where $p_a\geqq 6-2\epsilon$. (4) The surface formed of the pairs of points, one on each of two curves of genus 2. (*5) Surfaces on which the variable part of the canonical system is compounded with an involution of order 3 representable by a ruled surface of genus $\leqq\frac{1}{3}(p^{(1)}-3p_g+5)$ and also $\leqq\frac{1}{2}(p_g-p_a-2)$. (6) Surfaces on which the variable part of the canonical system is compounded with an involution of order 2 representable by a ruled surface of genus $\leqq\frac{1}{2}(p^{(1)}-2p_g+3)$. (*7) Same as (6) except that the ruled surface representing the involution is of genus p_g-p_a or p_g-p_a-1, and several other inequalities are necessary. *P. Du Val* (Athens, Ga.).

Jongmans, F. Sur l'étude des surfaces algébriques caractérisées par la condition $p_g\geq 2(p_a+2)$. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 485–494 (1950).

It is shewn that every algebraic surface whose geometric and arithmetic genera p_g, p_a satisfy $p_g\geqq 2(p_a+2)$ has on it two pencils of curves of genera $p_g-p_a-\epsilon$, ϵ respectively, the curves of the former being of genus θ, where $0\leqq\epsilon\leqq\theta\leqq 2$, and belongs to one of the following five classes: (a) ruled surface of genus $p>1$ ($p_g=0$, $p_a=-p$, $\epsilon=\theta=0$); (b) elliptic surfaces with $p_g>1$, $p_a=-1$ ($\epsilon=\theta=1$); (c) surfaces for which $p^{(1)}=1$, and $p_a\geqq 0$ ($p_g\geqq 4$, $\epsilon=0$, $\theta=1$); (d) surfaces representing pairs of points of two curves of genera 2, $h\geqq 2$ ($p_g=2h$, $p_a=h-2$, $\epsilon=\theta=2$); (e) surfaces (if any exist) for which $\epsilon=1$ or 0, $\theta=2$, and $p_a\geqq 4$, the variable part of the canonical system being compounded either with the first pencil, or with the canonical involution on the curves of this pencil. Except for this last possibility, which was overlooked by him, and which is not established by examples, the classification agrees with that of Comessatti [Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (5) **28**$_2$, 65–68, 123–127 (1919); Rend. Circ. Mat. Palermo **46**, 1–48 (1922)]. *P. Du Val* (Athens, Ga.).

Segre, Beniamino. Intorno agli spazi lineari situati sulle quadriche di un iperspazio. Univ. e Politecnico Torino. Rend. Sem. Mat. **9**, 137–144 (1950).

The condition that an S_k of S_r lies on a nonsingular quadric primal of S_r is expressed in terms of the fundamental Schubert conditions for S_k's of S_r. Furthermore, h general quadrics of S_r have a finite number m of S_k's in common if $r=\frac{1}{2}hk+h+k$, and a formula is given from which m could be calculated. In particular, the author gives the number of lines common to $2l$ general quadrics in S_{2l+1}, and the number of S_k's common to two general quadrics in S_{2k+2}. The second half of the paper establishes the results on the two systems of S_d's of a quadric in S_{2d+1} quoted without proof in a previous paper [Colloque de géometrie algébrique, Liège, 1949, pp. 123–142, Thone, Liège; Masson, Paris, 1950, pp. 137–138; these Rev. **12**, 200]. *D. B. Scott* (London).

Nollet, Louis. Sur les surfaces algébriques à système canonique réductible et sur quelques questions connexes. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **37**, 42–52 (1951).

By consideration of the everywhere finite simple integrals on the surface, it is shewn that if the variable part of the canonical system on an algebraic surface (of geometric and arithmetic genera p_g, p_a) is compounded with a pencil of genus x, of curves of genus p, then either: (1) the surface is elliptic, $p_g=x\geqq 2$, $p_a=-1$, $p=1$; or (2) $p_g\geqq 2$, $p_a\geqq 0$, $x=p_g-p_a$, $1\leqq p\leqq 5$, and if $p>1$, then $p_a>0$. It is deduced as a corollary from this that if the absolute linear genus of a surface is $p^{(1)}>1$, then the bicanonical system is regular, unless $p_a=0$ and $p_g=0$ or 1. As a further corollary if a surface has $p^{(1)}>1$, $p_a\geqq 0$, and $p_g\geqq 2(p_a+2)$, the variable part of the canonical system is irreducible. *P. Du Val.*

d'Orgeval, Bernard. Observations sur une note de M. Nollet. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **9**, 65–67 (1950).

Nollet [Acad. Roy. Belgique Bull. Cl. Sci. (5) **35**, 1016–1026 (1949); these Rev. **11**, 538] shewed that the irregularity $q=p_g-p_a$ of a surface whose canonical system is compounded with a pencil, of positive grade, with or without a fixed part, cannot exceed 4. The present paper strengthens this result to $q\leqq 3$. *P. Du Val* (Athens, Ga.).

Roth, Leonard. Sulle V_3 algebriche su cui l'aggiunzione si estingue. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **9**, 246–250 (1950).

The well-known results of Castelnuovo and Enriques [see O. Zariski, Algebraic Surfaces, Springer, Berlin, 1935] to the effect that if an algebraic surface F carries a linear system $|A|$ such that the process of successive adjunction when applied to $|A|$ terminates in a finite number of steps, then F is birationally equivalent to a ruled surface, is

generalized to the case of completely regular 3-dimensional varieties V_3. If V_3 is completely regular and carries such a system $|A|$ of surfaces, then it is shown by examining 16 different cases that, with possibly two exceptions, V_3 is either a rational or a unirational variety. The possible exceptions are (1) the general quartic hypersurface of S_4 and (2) a double S_3 with a general surface of order 6 as branch variety. The analysis is carried out by examining the possible values of the virtual characters of $|A|$ which satisfy the numerical conditions imposed by the termination of the adjunction process. As a final remark it is pointed out that an irregular V_3 presents no new cases if $|A|$ is regular, but new types (of which no examples are as yet known) do arise if $|A|$ is irregular. *H. T. Muhly.*

Okugawa, Kôtaro. Linear conditions at a point. Mem. Coll. Sci. Univ. Kyoto Ser. A. **25**, 99–102 (1949).

Okugawa, Kôtaro. Base conditions for hypersurfaces at a point. J. Math. Soc. Japan **1**, 242–250 (1950).

In the first paper, various assertions are made concerning linear conditions at a point of n-space, that is, linear conditions on the coefficients of polynomials or power series in the variable coordinates in the space which vanish at the point in question. In the second paper, these assertions are recast into the language of linear functionals, and proofs are given. Let L denote the ring of formal power series in n variables over a field K, topologized in the usual fashion by taking the powers of the maximal ideal as a fundamental system of neighborhoods of zero. Let Λ denote the space of all continuous linear mappings of L into K, whereby K is taken discrete; let Λ also be assigned discrete topology. It is proved that the customary duality relations hold between the closed linear subspaces of L and the linear subspaces of Λ, corresponding subspaces being annihilators of each other. The space Λ is now construed as the Macaulay inverse system in the following fashion: Let $\{m_i\}$ be an enumeration of the monomials in L, let ξ_i be the continuous linear function on L such that $\xi_i(m_j)=\delta_{ij}$; the ξ_i form a basis of Λ. If $f\in L$ and $\phi\in\Lambda$, a product $f\times\phi$ in Λ is defined by the requirement that $f\times\phi$ be additive and K-linear in each argument, together with the formula $m_i\times\xi_j=\xi_k$ if $m_j=m_im_k$, $m_i\times\xi_j=0$ if m_j is not a multiple of m_i. It is then easily proved that Λ thus becomes an L-module and that $\phi(fg)=g\times\phi(f)$ for $f, g\in L$, $\phi\in\Lambda$, whence it follows that the dual pairing described above is one-to-one between the ideals of L and the L-submodules of Λ. Various consequences are then obtained; for example, an ideal in L is 0-dimensional and intersection-irreducible if and only if its annihilator is a principal L-submodule. *I. S. Cohen* (Cambridge, Mass.).

Gröbner, W. Sulle varietà perfette. Ann. Mat. Pura Appl. (4) **28**, 217–219 (1949).

The author discusses the notion of a perfect polynomial ideal and its relationship to Dubreil's ideals of the first species and to the generalized theorem of Noether. *I. S. Cohen* (Cambridge, Mass.).

Chow, Wei-Liang. On the defining field of a divisor in an algebraic variety. Proc. Amer. Math. Soc. **1**, 797–799 (1950).

Let U be an algebraic variety of dimension r and let G be a d-cycle on U, i.e., an integral linear combination of simple subvarieties of U of dimension d [for definitions, cf. Weil, Foundations of Algebraic Geometry, Amer. Math. Soc. Colloq. Publ., v. 29, New York, 1946; these Rev. **9**, 303]. The question considered is whether there always exists a smallest field over which G is rational. A simple counter-example is given to show that the answer is in the negative. On the other hand, the author proves a conjecture of Weil to the effect that such a least field exists for every divisor ($(r-1)$-cycle) on V. This field is obtained by adjoining to the field of definition of U the ratios of the coefficients of the associated form of G. *I. S. Cohen.*

Northcott, D. G. The number of analytic branches of a variety. J. London Math. Soc. **25**, 275–279 (1950).

Let V be an irreducible algebraic variety, W an irreducible subvariety, Q the quotient ring of W on V. It is known that the zero ideal of the completion of Q is an irredundant intersection of prime ideals; the number h of ideals occuring in this intersection is called the number of analytic branches of V at W. The principal theorem proved is that this number equals the number of maximal ideals in the integral closure of Q. The proof makes use of, and the theorem generalizes, Zariski's theorem on the analytical irreducibility of normal varieties [Ann. of Math. (2) **49**, 352–361 (1948); these Rev. **9**, 460]. As a consequence it follows easily that in any birational transformation of V in which W is not fundamental, there are at most h subvarieties of the transform corresponding to W. *I. S. Cohen* (Cambridge, Mass.).

Northcott, D. G. An application of local uniformization to the theory of divisors. Proc. Cambridge Philos. Soc. **47**, 279–285 (1951).

Démonstration, en s'en tenant strictement aux méthodes de A. Weil [Foundations of algebraic geometry, Amer. Math. Soc. Colloq. Publ., v. 29, 1946; ces Rev. **9**, 303] du fait que l'anneau local d'un point simple d'une variété algébrique V^n est un anneau à unique factorisation. Ceci avait été démontré par Zariski, par des méthodes "analytiques" [Trans. Amer. Math. Soc. **62**, 1–52 (1947); ces Rev. **9**, 99]. L'outil utilisé ici est la variété W^n de point générique $(x, F_i(x))$, où (x) est point générique de V^n et où les $F_i(X)$ sont n formes linéaires uniformisantes pour V en un point simple de V. Le résultat principal est: soient V^n une variété définie sur le corps infini k, Z un diviseur positif de V rationnel sur k, $P_0, \cdots, P_m$ des points simples de V; il existe un polynome $G(X)$ sur k découpant sur V le diviseur $Z+Z'$, où Z' est positif, sans composante commune avec Z, et ne contient aucun P_j. *P. Samuel.*

Abellanas, Pedro. Théorie arithmétique des correspondances algébriques. Revista Mat. Hisp.-Amer. (4) **9**, 175–233 (1949).

Let T be an irreducible algebraic correspondence between the varieties V and V' over the ground field k, let $(\xi_0, \cdots, \xi_n; \eta_0, \cdots, \eta_m)$ be a bihomogeneous general point of T, let $\Sigma=k(\xi_0, \cdots, \xi_n)$, $\Sigma'=k(\eta_0, \cdots, \eta_m)$,

$$\Omega=k(\xi_0, \cdots, \xi_n; \eta_0, \cdots, \eta_m).$$

Following Zariski's definition for the case of a birational correspondence [Trans. Amer. Math. Soc. **53**, 490–542 (1943); these Rev. **5**, 11], the author calls irreducible subvarieties W and W' of V and V' respectively corresponding if there exists a valuation of the field Ω such that the induced valuations of Σ and Σ' have as centers on V and V' the subvarieties W and W'. In order to use this definition effectively, some preliminary material on valuation theory is needed, and part I of the paper is devoted in the main to results on the composition of valuations of algebraic function fields and to extension of valuations to extensions of such fields, particular regard being paid to the centers of

the valuations. Also included in part I is a discussion of varieties in biprojective space.

Part II begins with the definition of corresponding subvarieties given above. Now let $P=k[\xi_0, \cdots, \xi_n]$,

$$P'=k[\eta_0, \cdots, \eta_m], \qquad P^*=k[\xi_0, \cdots, \xi_n; \eta_0, \cdots, \eta_m].$$

Let $\mathfrak{P}$ be the homogeneous prime ideal of W in $P, \mathfrak{P}_1^*, \mathfrak{P}_2^*, \cdots$ the minimal prime ideals of $P^*\mathfrak{P}$ which contract to $\mathfrak{P}$, $\mathfrak{P}'_i=\mathfrak{P}_i^* \cap P'$. Then the subvarieties of V' defined by the $\mathfrak{P}_i'$ all correspond to W, and every subvariety of V' corresponding to W lies on one of these. The correspondence T induces in a natural way an algebraic correspondence between the derived normal models of V and V', and the properties of T can be read off from this new correspondence. Hence, let it now be assumed that V and V' are normal. It is proved that the degree of transcendence (d.t.) of $P^*/\mathfrak{P}_i^*$ over $P/\mathfrak{P}$ is greater than or equal to the d.t. of Ω over Σ; if equality holds for each i, the variety W is defined to be nonfundamental for the correspondence T. A definition in similar terms is given for W being regular for T. Various criteria are then given for W to be nonfundamental or regular, some of them in terms of a sort of parametric representation for the correspondence. There is also a result connecting the dimension of a regular W with that of a corresponding W'. *I. S. Cohen* (Cambridge, Mass.).

Bagchi, Haridas. Note on a certain Cremona transformation associated with a plane triangle. Rend. Sem. Mat. Univ. Padova **19**, 231–236 (1950).

Verfasser beschreibt eine ebene quadratische Transformation, die sich in baryzentrischen Koordinaten bezüglich eines gegebenen Dreiecks A, B, C durch die Formeln:

$$\rho x'=1/y+1/z, \qquad \rho y'=1/x+1/z, \qquad \rho z'=1/x+1/y$$

beschreiben lässt. Bei der Umkehrung T^{-1} ist A, B, C das Fundamentaldreieck, bei T das der Seitenmitten. Der Schwerpunkt von A, B, C erweist sich als Fixpunkt von T. *W. Burau* (Hamburg).

Benedicty, Mario. Sopra una trasformazione cremoniana collegata con la teoria delle funzioni quasi abeliane. Ann. Scuola Norm. Super. Pisa (3) **4**, 27–33 (1950).

Con riferimento a ricerche di F. Severi [Funzioni quasi abeliane, Pontificiae Academiae Scientiarum, Scripta Varia, v. 4, 1947; questi Rev. **9**, 578] e del recensore [Ann. Mat. Pura Appl. (4) **27**, 273–291 (1948); questi Rev. **11**, 391], l'autore studia la trasformazione cremoniana, che intercorre tra due spazi proiettivi considerati come due modelli della varietà quasi abeliana di Jacobi di un campo neutro di caratteri $p=0$, $\pi=\delta=\delta_1+\delta_2$. La trasformazione tra tali modelli S_δ, S_δ' è rappresentata in coordinate omogenee dalle:

$$x_i/x_0=\beta_i(y)/\alpha_i(y) \qquad (i=1, 2, \cdots, \delta),$$

ove, in S_δ, $\alpha_i=0$ $(i=1, 2, \cdots, \delta)$, $\beta_j=0$ $(j=1, 2, \cdots, \delta_1)$ sono gli iperpiani di infinito e $\alpha_i=\beta_i=0$ $(i=\delta_1+1, \cdots, \delta)$ gli $S_{\delta-2}^{**}$ di indeterminazione degli integrali normali virtualmente di prima specie; in S_δ' gli elementi analoghi sono $x_j=0$, $x_0=0$ e gli $S_{\delta-2}^{*}$, $x_i=x_0=0$. La trasformazione è di ordine δ e i sistemi omaloidici sono dati dalle forme di ordine δ passanti, in S_δ, per gli $S_{\delta-2}^{**}$ e per gli $S_{\delta-2}^{*}$ coordinati per l'origine; in S_δ', per gli $S_{\delta-2}^{*}$ e per una certa varietà irriducibile $V_{\delta-2}^{\frac{\delta^2-3\delta+2}{2}}$.

Si studiano gli elementi fondamentali ed eccezionali della corrispondenza, tra i quali hanno particolare interesse, per il loro legame con la teoria delle funzioni quasi abeliane, gli $S_{\delta-2}^{**}$ di S_δ, immagini dei gruppi contenenti coppie neutre, e le rette ad essi appoggiate, immagini delle g_δ^1 neutre speciali; queste ultime si mutano nei punti della $V_{\delta-2}$, di modo che S_δ' rappresenta in maniera generalmente biunivoca le g_δ neutre (speciali e non speciali) del campo. *F. Conforto.*

Differential Geometry

Vivanti, G. Les courbes planes à normales doubles. Enseignement Math. **39** (1942–1950), 46–49 (1951).

Viguier, Gabriel. Les développantes généralisées du second ordre d'une courbe plane. Ann. Fac. Sci. Univ. Toulouse (4) **12**, 83–89 (1950).

This is an amplified version of an earlier note [C. R. Acad. Sci. Paris **229**, 462–464 (1949); these Rev. **11**, 393]. The analytical work is given in greater detail, and some additional particular cases are discussed. *L. A. MacColl.*

Viguier, Gabriel. Canonisation géométrique spatiale de l'équation de Riccati. Ann. Fac. Sci. Univ. Toulouse (4) **12**, 77–82 (1950).

Two parameterized curves, a base curve Γ_1 and an adjoint curve Γ_2, are given in three-dimensional space. Let $M(t)$ and $L(t)$ denote the points of Γ_1 and Γ_2 which correspond to the same typical value of the parameter. It is required to determine a function $\tau(t)$ such that if a vector of length $\tau(t)$ is laid off from $M(t)$ on the tangent to Γ_1, the extremity of the vector describes a curve Γ_2 whose tangent, at the point corresponding to t, passes through $L(t)$. [It is clear that further conditions are necessary in order to insure that a solution exists. However, the author makes no mention of these.] It is shown that if a solution exists, $\tau(t)$ is a solution of a certain Riccati equation. Conversely, it is shown that if a Riccati equation and a base curve Γ_1 are given, an adjoint curve Γ_2 can be found such that the two curves are related to the equation in the manner described above. One special situation is discussed in detail. *L. A. MacColl.*

Manikarnikamma, S. N. The spherical Bertrand curve. Math. Student **18** (1950), 22–24 (1951).

Deweck, M. Courbes tracées sur une surface développable. Mathesis **60**, 84–88 (1951).

Kahane, Arno. Sur les propriétés centro-affines des courbes planes et gauches. Acad. Repub. Pop. Române. Stud. Cerc. Mat. **1**, 175–250 (1950). (Romanian. Russian and French summaries)

A linear homogeneous ordinary differential equation of order n defines through its solutions a class of skew curves in n-space, all centro-affinely equivalent. The author takes up a detailed study of the cases $n=2, 3$. Analyticity is assumed. Among the topics discussed are: equivalence, invariants, canonical forms, dual forms, geometrical characterizations, special forms integrable in terms of known functions. The methods are generally elementary, the results richly varied. *J. L. Vanderslice* (College Park, Md.).

Galafassi, Vittorio Emanuele. Di una geometria differenziale simile nello studio delle curve piane. Univ. e Politecnico Torino. Rend. Sem. Mat. **9**, 297–308 (1950).

This is a study of the differential geometry of plane curves in Euclidean (as opposed to Euclidean-metric) space. For example, the first finite invariant analogous to curvature is

of the third order. Relations with projective, affine, and Euclidean-metric geometry are constantly in the foreground. The presentation is pleasantly elementary and copiously annotated. *J. L. Vanderslice* (College Park, Md.).

Roşca, R. Sur une classe de réseaux autoprojectifs. Acad. Repub. Pop. Române. Stud. Cerc. Mat. **1**, 169–174 (1950). (Romanian. Russian and French summaries)

This paper proves three related theorems in the differential geometry of nets (Tzitzeica). One of them is the following. If a net (x) has two of the following properties it likewise has the third: (1) it is an R-net; (2) it is projective with its second Laplace transform; (3) three of its Laplace transforms of equal parity are collinear.

J. L. Vanderslice (College Park, Md.).

Mineo, Corradino. Superficie sulle quali il triangolo geodetico ha un semplice grado di mobilità. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **9**, 213–216 (1950).

It was proved by Mangoldt [J. Reine Angew. Math. **94**, 21–40 (1883)] that the only surfaces upon which the geodesic triangle has only one degree of freedom are surfaces applicable to a surface of rotation. The result was obtained by Mangoldt by long and laborious calculations. Darboux substituted a geometrical proof which is open to some objections. The author in this note succeeds in giving a simple proof of the above result. *E. T. Davies* (Southampton).

Tortorici, Maria. Sulle trasformazioni asintotiche delle curve. Boll. Un. Mat. Ital. (3) **6**, 22–27 (1951).

The author states the condition that two curves C and C_1 be asymptotic on the ruled surface R swept out by the lines joining corresponding points of C and C_1, and then considers in a simple, direct way the special cases where C_1 is an orthogonal trajectory of the generators of R, and where C and C_1 have the same torsion at corresponding points. Short proofs of some of the results of Bianchi [Rend. Circ. Mat. Palermo **25**, 291–325 (1908)] and Picone [ibid. **39**, 51–73 (1915); Note e Memorie Matematica. Circ. Mat. Catania **1**, 187–197 (1921)] are presented, including a proof of the theorem that a ruled surface is a helicoid of minimum area if it contains more than two asymptotic curves with constant torsion. *A. Schwartz* (New York, N. Y.).

Hartman, Philip, and Wintner, Aurel. On the asymptotic curves of a surface. Amer. J. Math. **73**, 149–172 (1951).

It is shown that on a surface S in Euclidean 3-space of class C'', and having a C''-parametrization in which the Gaussian curvature K satisfies a Lipschitz condition, either of the asymptotic directions at a point P where $K<0$ determines a unique asymptotic curve through P. If S has constant negative curvature, it need only be of class C'' for the above conclusion to obtain. However, for an arbitrary S of class C'', with $K<0$ at P, it is possible for a continuum of asymptotic curves to issue from P. Asymptotic parametrizations exist near every point of an S of negative curvature K, possessing a C''-parametrization in which K is of class C'. In particular, if K is constant, then a Tchebychef parametrization exists. Results analogous to those derived for asymptotic lines are proved for lines of curvature. Analogues of all these theorems are obtained for torses.

A. Fialkow (Brooklyn, N. Y.).

Hartman, Philip, and Wintner, Aurel. On the problems of geodesics in the small. Amer. J. Math. **73**, 132–148 (1951).

The authors prove that if the coefficients $g_{ij}(u, v)$ of the positive definite line element of a surface are of class C' in the vicinity of $(u, v)=(0, 0)$, then to every circle C_ϵ about $(0, 0)$ of sufficiently small radius ϵ, there exists a circle C_δ about $(0, 0)$ such that every point of C_δ can be joined with $(0, 0)$ by a geodesic contained in C_ϵ. Here "geodesic" means a solution of the defining second order differential equations. A number of questions concerning the above situation are asked and answered of which the following are typical: (i) Is the geodesic supplied by the theorem unique? (ii) Is the minimizing arc unique? (iii) Must a geodesic furnish a minimal distance between some pair of points on it? It is shown that under the assumptions of the above theorem, the answer to each question is in the negative, but becomes affirmative if the partial derivatives of the g_{ij} satisfy a uniform Lipschitz condition. *A. Fialkow.*

Wunderlich, Walter. Raumkurven, die pseudogeodätische Linien eines Zylinders und eines Kegels sind. Compositio Math. **8**, 169–184 (1950).

A curve on a surface Φ in Euclidean three-space is called a "pseudo-geodesic" of Φ if at every point of k the angle between the tangent plane of Φ and the osculating plane to k has the same value γ. This paper is concerned with those curves k which are simultaneously pseudo-geodesic of some cylinder Φ_1 and some cone Φ_2 (vertex O). Let l be the transform of k by means of a polarization with respect to a sphere S with O as center. Then l is a loxodrome on the cone Ol. Moreover, the polar cone of the cone Ol intersects S in a spherical loxodrome. The author uses these properties to study l and k, respectively, and the line congruences obtained by revolving the tangents to l and k, respectively, about the line through O and parallel to the generators of Φ_1. No curve exists which is pseudogeodesic of two cylinders.

W. van der Kulk (Providence, R. I.).

Wunderlich, Walter. Pseudogeodätische Linien auf Zylinderflächen. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **158**, 61–73 (1950).

A curve k on a surface Φ in Euclidean three-space is called a "pseudo-geodesic" of Φ if at every point of k the angle between the tangent plane to Φ and the osculating plane to k has the same value γ. In this paper Φ is assumed to be a cylinder. It is shown that the image on the unit sphere of the tangents to k is a spherical loxodrome dependent only on γ and on the direction δ of the generators of Φ. The author studies the curve $\bar{k}$ obtained from k by means of a polarisation with respect to a paraboloid of revolution with axis parallel to δ. Special attention is given the case where Φ is a circular cylinder. Among the many properties of k in that case we mention the following: k is the equilibrium position of a homogeneous chain suspended from two points of Φ and subject to an ordinary gravitation in the direction of δ (Φ is assumed smooth). *W. van der Kulk.*

Wunderlich, Walter. Pseudogeodätische Linien auf Kegelflächen. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **158**, 75–105 (1950).

A curve on a surface Φ in Euclidean three-space is called a "pseudo-geodesic" of Φ if at every point of k the angle between the tangent plane to Φ and the osculating plane to k has the same value γ. In this paper Φ is assumed to be a cone (vertex of Φ is P). Let $\bar{k}$ be the transform of k by means

of a polarization with respect to a sphere with P as center. Then k is a pseudo-geodesic if, and only if, $\bar{k}$ is a loxodrome on the cone $P\bar{k}$. Special attention is given the case where Φ is a circular cone. In that case k is a screw line on some quadratic surface of revolution Δ coaxial with Φ. Depending as to whether Δ is an ellipsoid, hyperboloid, paraboloid or cone the author distinguishes four types of curves k. Among the many properties of k when Φ is a circular cone we mention the following: k is the equilibrium position of a homogeneous chain suspended from two points of Φ and subject to a Newtonian gravitation with P as center (Φ is assumed smooth). *W. van der Kulk* (Providence, R. I.).

Borisov, Yu. F. Manifolds of bounded curvature with a boundary. Doklady Akad. Nauk SSSR (N.S.) **74**, 877–880 (1950). (Russian)

On surfaces without boundary the existence of an angle [all concepts are used in the sense of A. D. Aleksandrov, same Doklady (N.S.) **60**, 1483–1486 (1948); these Rev. **10**, 147] between the curves L_1, L_2 issuing from z is equivalent to the existence of, and equal to, the limit of the angle between two geodesic chords zx_i, where $x_i \in L_i$ and $x_i \to z$. If the surface has boundaries, then the existence of the angle implies the existence of the limit and the two are equal, but not conversely. If the total geodesic curvatures of the chords zx_i tend to 0, the existence of the limit implies that of the angle. If L_1 and L_2 bound a sector U and if the sector angle exists, then (under an additional, geometrically obvious condition) the angle between L_1 and L_2 exists and equals the sector angle. In terms of angles, conditions are given under which a piece of a general manifold is isometric to a manifold with bounded curvature. Also, necessary and sufficient conditions are given under which gluing together of manifolds with bounded curvature yields a manifold of bounded curvature. *H. Busemann* (Los Angeles, Calif.).

Terracini, Alessandro. Direttrici congiunte di una rigata. Univ. e. Politecnico Torino. Rend. Sem. Mat. **9**, 325–342 (1950).

Dans un ordre d'idées analogue à celui développé dans une note antérieure [Ann. Mat. Pura Appl. (4) **30**, 267–275 (1949); ces Rev. **12**, 129], l'auteur montre comment on peut définir, sur la surface réglée (R) de l'espace ordinaire la plus générale, un système de couples de courbes dont le lien avec la surface est invariant par applicabilité projective. $y(t)$, $z(t)$ étant deux points d'une génératrice (yz) de R, et C_y, C_z deux directrices quelconques de R supposées non asymptotiques, il existe un complexe linéaire spécial H (de directrice yz) contenant les faisceaux de droites tangentes à R en y, z et aux points respectivement infiniment voisins sur C_y, C_z; de même, si η, ζ sont les plans tangents à R en y et z, il existe un complexe linéaire K déterminé par les faisceaux $y\zeta$, $z\eta$ et les faisceaux analogues ayant pour centres les points infiniment voisins de y et z sur C_y et C_z. D'autre part les bandes formées par les points de C_y (ou C_z) et les plans tangents correspondants à R, représentées sur la quadrique M_4^3 de Klein de S_5, fournissent deux surfaces réglées (g_y), (g_z) se correspondant par génératrices g_y, g_z coplanaires. Si dès lors on envisage le plan π de deux génératrices homologues, on trouve que ce plan engendre un système ∞^1 tel que chaque plan soit incident avec le plan infiniment voisin, le point x caractéristique de π (distinct du point de contact de π avec M_4^3) étant précisément l'image de K dans S_5.

L'incidence mise en évidence peut avoir lieu, comme l'auteur l'a montré antérieurement [Sull'incidenza di spazi infinitamente vicini, Scritti mat. offerti à Luigi Berzolari, pp. 449–478, Pavia, 1936], suivant différents ordres d'approximation σ [σ = entier pair, tel que $2 \leqq \sigma \leqq 16$ dans le cas général, et $\sigma = \infty$ si les plans sont deux à deux incidents]. Ce rapprochement amène à considerer les couples C_y, C_z d'une R non développable pour lesquels, non seulement les faisceaux de droites tangentes à R en y et z et en un couple de points infiniment voisins sur C_y et C_z appartiennent à un même complexe linéaire H, mais où cette appartenance, qui dans le cas général a lieu avec l'ordre d'approximation $\sigma = 2$, ait lieu avec un ordre $\sigma \geqq 4$. L'auteur appelle conjointes sur R deux directrices C_y, C_z pour lesquelles $\sigma \geqq 4$, biconjointes deux telles courbes pour lesquelles $\sigma \geqq 6$, et il étudie les conditions, tant analytiques que géométriques pour que deux courbes C_y, C_z de R répondent à l'une ou l'autre condition. Les résultats sont remarquables. Pour les couples conjoints le résultat principal est le suivant: C_z n'étant ni asymptotique ni branche flecnodale, elle a ∞^1 conjointes définies par une équation de Riccati. Si C_z est branche (non rectiligne) flecnodale, les C_y se réduisent à la $2^{\text{ième}}$ branche flecnodale et n'existent pas si les deux branches flecnodales sont confondues. Les ∞^1 conjointes à une C_z (non flecnodale) le sont aussi à ∞^1 directrices de R, dites associées à C_z, constituant sur R un système homographique, et susceptibles d'une définition indépendante de la notion de courbes conjointes: Pour que les courbes d'un système ∞^1 de R soient associées, il faut et il suffit que leurs tangentes aux points d'une même génératrice quelconque g de R forment une demi-quadrique dont la complémentaire soit contenue dans la congruence linéaire osculatrice à R le long de g. L'étude des couples biconjoints de directrices de R conduit à envisager les deux cas où R appartient ou non à un complexe linéaire. Dans le $2^{\text{ième}}$ cas on a ∞^3 couples biconjoints, auxquels viennent s'ajouter, dans le 1^{er} cas, les couples de directrices (non asymptotiques) conjuguées harmoniques par rapport aux deux asymptotiques de Lie. Pour terminer l'auteur montre, comme il a été dit au début, que les notions de directrices conjointes ou biconjointes sont invariantes par déformation projective, et il caractérise géométriquement les deux cas au moyen d'une propriété du complexe K précédemment défini ou de certains complexes quadratiques. *P. Vincensini* (Marseille).

Longo, Carmelo. Studio numerativo sopra le varietà di contatto delle superficie in uno spazio ad n dimensioni. Ann. Scuola Norm. Super. Pisa (3) **4**, 223–230 (1950).

L'auteur étudie la variété $W(k, h)$, lieu des S_h osculateurs aux courbes d'une surface S passant par un E_k donné. Cette étude a été inaugurée par Del Pezzo [Rend. Accad. Sci. Fis. Mat. Napoli **25**, 176–180 (1886)], qui a déterminé l'ordre des $W(2, 0)$ pour une surface ou une V_m. C. Segre a complété le résultat précédent par la détermination de l'ordre des $W(3, 0)$ pour une surface. La question a été reprise par Bompiani [Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **48**, 393–410 (1913)]. Bompiani a largement étendu la recherche, déterminant pour une V_m les ordres des $V(3, 0)$, $W(4, 1)$, et des $W(k, k-3)$ pour $k > 3$, et donnant les directives générales relatives à la détermination des ordres des $W(k, h)$, problème dont B. Levi, L. A. Santaló, et C. De María se sont à leur tour occupés dans un travail [Fac. Ci. Mat. Univ. Nac. Litoral. Publ. Inst. Mat. **8**, 3–72 (1946); ces Rev. **9**, 62]. Une presomption de F. Severi, suivant laquelle, pour une surface, l'ordre d'une $W(k, 0)$ de dimension $2k$ serait $(2k)!/k!(k+1)!$ a amené l'auteur à revenir sur le sujet, et à rectifier une inexactitude qui s'était glissée dans le travail de Levi, Santaló, et C. De María dans la déter-

mination des ordres des variétés $W(6, 1)$ et $W(6, 0)$, ordres qui, au lieu de 12 et 124, sont respectivement 10 et 132.

P. Vincensini (Marseille).

Reade, Maxwell O. A characterization of minimal surfaces in isothermic representation. Proc. Amer. Math. Soc. **2**, 47–54 (1951).

For sufficiently smooth real functions $x_j \equiv x_j(u, v) \equiv x_j(z)$, $j=1, 2, 3$, the author considers the expression

$$\Phi(x, z, r) \equiv \sum_{j=1}^{3} \left[\int\int_{|\zeta - z| < r} (\zeta - z) x_j(\zeta) d\xi d\eta \right]^2, \qquad \zeta = \xi + i\eta,$$

noting that Φ generalizes a function considered by Federoff [C. R. Acad. Sci. Paris **193**, 512–513 (1931)], and that Φ is comparable with another integral expression which the author and the reviewer have studied [Trans. Amer. Math. Soc. **49**, 354–377 (1941); these Rev. **2**, 360]. It is shown that conformal maps on spheres of finite radius are characterized by $\Phi = o(r^8)$, $\Phi \neq o(r^8)$, and that conformal maps on minimal surfaces are characterized by $\Phi = o(r^8)$. Further, the point condition $\Phi = o(r^8)$ implies the finite condition $\Phi = 0$. As the author observes, it would be interesting to obtain an extension to the characterization of conformal plane maps.

E. F. Beckenbach (Los Angeles, Calif.).

Efremovič, V. A. Infinitesimal spaces. Doklady Akad. Nauk SSSR (N.S.) **76**, 341–343 (1951). (Russian)

The outlined program would be of great interest if carried out in detail. Unfortunately this note is still almost as vague as the earlier brief report [Uspehi Matem. Nauk (N.S.) **4**, no. 3(30), 178 (1949); these Rev. **11**, 195].

H. Busemann (Los Angeles, Calif.).

Kurita, Minoru. One parametric motion in Klein spaces. Nagoya Math. J. **1**, 19–23 (1950).

In the Euclidean plane, one-parametric motion is in general a motion in which a certain curve rolls on another fixed curve without slipping. In the present paper this fact is extended to the case of Klein spaces. Generalizations of the notions of "instantaneous centre of motion" and "rolling without slipping" are defined and a theorem analogous to the above is given. If S is the one-parametric subgroup of the fundamental group G and H a subgroup of G, then the rolling figures are generated by a one-parametric set of cosets of H. Applications to the spherical, Euclidean, and projective motions are given. In the last case we find in general two figures of one-parametrical sets of $n+1$ points with a common set of invariants rolling along each other.

J. A. Schouten (Epe).

***van Oosten, Cornelis Petrus Stephanus. Meetkundige beschouwingen over bewegingen in een Euclidische ruimte. [Études géométriques sur les mouvements dans un espace euclidien].** Thesis, University of Leiden, P. Noordhoff, Groningen-Djakarta, 1951. viii+99 pp. (Dutch. French and Esperanto summaries)

A general rigid motion in a plane is completely defined by two loci: The locus P_1 of instantaneous centers of rotation and the locus P_2 of points which become instantaneous centers of rotation. The author generalizes this known fact for a rigid motion in a space of $m \geqq 2$ dimensions. The basic idea may be described as follows: Let $x^\alpha(\zeta^\lambda)$ be the coordinates of a point in the fixed (moving) space. Then (1) $x^\alpha = B^\alpha{}_\lambda \zeta^\lambda + b^\alpha$ where $B^\alpha{}_\lambda$, b^α are functions of time and $B^\alpha{}_\lambda$ are coefficients of an orthogonal transformation ($B^\alpha{}_\lambda = B^\lambda{}_\alpha$). From (1) one obtains (2) $v^\alpha = dx^\alpha/dt = p^\alpha{}_i x^i - p^\alpha{}_i b^i + db^\alpha/dt$ ($p^\alpha{}_i = -p^i{}_\alpha = B^\alpha{}_\lambda dB^\lambda{}_i/dt$) and consequently the points which at a given time are fixed satisfy (3) $p^\alpha{}_i x^i - p^\alpha{}_i b^i + db^\alpha/dt = 0$. Since for $m = 2n+1$ the determinant of $p^\alpha{}_i$ is equal to zero (and for $m = 2n$ is a complete square) these cases of (2) have to be dealt with separately. The author describes completely all four cases of $m = 2n+1$, $m = 2n$ with or without fixed point supposing that the rank of the determinant of $p^\alpha{}_i$ is the highest possible one. Thus, for instance, in the case $m = 2n$ without a fixed point the configuration of the loci corresponding to P_1 (and P_2) consists of a curve and $n-1$ ruled surfaces through it in the fixed space (in the moving space). The curves roll on each other without sliding and the corresponding ruled surfaces have a common ruling at any given time. Moreover, the author considers special motions for which the instantaneous velocity and acceleration have the same direction.

V. Hlavatý.

Il'in, I. G. On the rigidity of the lateral surface of a closed cylindroid. Uspehi Matem. Nauk (N.S.) **5**, no. 6(40), 145–156 (1950). (Russian)

Let S be a cylindrical surface in E^3, that is a surface consisting of a lateral surface B of class C^2 with vanishing Gauss curvature, homeomorphic to an ordinary cylindrical belt and of two pieces P_1, P_2 of parallel planes whose boundary curves do not contain straight segments. The lateral surface B is infinitesimally deformable if and only if the area bounded by the (plane) curve formed by the midpoints of the segments forming B (which is developable) equals a fourth of the sum of the areas of P_1 and P_2. This generalizes a previous result of Efimoff [Mat. Sbornik N.S. **20(62)**, 27–53 (1947); these Rev. **8**, 597], who showed that B is not deformable for convex S. Examples are given to show that the case of deformability really occurs.

H. Busemann.

Jonas, Hans. Ein mit der Verbiegung des Rotationsparaboloids verknüpftes Stratifikationsproblem. Math. Nachr. **5**, 39–68 (1951).

Starting from polar reciprocal surfaces the author deals with a great number of problems which are interesting from the classical point of view. The six sections deal with the following topics. 1. Connected pairs of surfaces. 2. Mutually stratifiable polar reciprocal congruences. 3. The case of the real rotational paraboloid. 4. Rotation paraboloid of Weingarten and its corresponding W-connection. 5. Line-coordinates; focal surfaces of the axial congruences of the W-connection. 6. The W-connection as a cycle of R-congruences; equivalent systems generated by stratification.

J. A. Schouten (Epe).

***Finikov, S. P. Teoriya kongruèncil. [Theory of Congruences].** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 528 pp.

Back in the thirties the author had conceived the idea of writing a monograph on the theory of congruences, based on the classical investigations of Bianchi, Ribaucour, and Guichard. The war intervened, and when after 1945 the author wanted to finish his work for publication, he realized that it was necessary to revise it thoroughly because of the many new results obtained, mainly by Russian authors. Especially mentioned are Bahvalov, Rossinskiĭ, Laptev, Rozenfel'd, Vasil'ev, Korovin, Koz'mina, and Smirnov. The author, who himself has made distinguished contributions to this field, has therefore shortened the more classical parts of the theory, which have found extensive treatment in the

later editions of Bianchi's "Lezioni di geometria differenziale," v. II [Spoerri, Pisa, 1923], and devoted more space to recent investigations. Among the authors whose work is included we find Bompiani, Vincensini, Jonas, Su, Terracini, and Haack. The result is a beautiful monograph which will go far in giving the theory of line congruences an equal standing with the theory of surfaces, to which it has often been added as a kind of afterthought.

The subject matter is the metrical and projective theory of line congruences and allied systems in three-dimensional space. After an introduction in the first chapter into the elementary Kummer-Hamilton theory, presented in vector analysis, the Cartan method is introduced in the second chapter. Use of this method throughout the book allows a uniform treatment of the varied material. The third and fourth chapters give the metrical theory of congruences as an instrument for the transformation of surfaces. After the selection of canonical systems of reference the author investigates pseudospherical congruences, W-congruences, Ribaucour congruences and cyclical systems. Transformations of Moutard lead to the theorems of permutability.

The fifth and sixth chapters discuss the theory of the asymptotic representations of surfaces in projective space. Here we find Fubini's theory of W-congruences, and the theory of conjugate nets in n-dimensional projective space P_n based on Laplace's equation. The next two chapters give the theory of congruences in line space with the aid of the representation of lines on a hypersurface Φ_4^3 in P_5. Here is also a metrical theory, which leads to parallel and orthogonal nets.

The ninth chapter deals with invariants of congruences in P_3 and is primarily based on lithographic notes of Cartan's lectures. The next two chapters give applications of this theory to congruences-R (in a net-R both congruences are W-congruences) and congruences-Φ (here the lines of Darboux correspond on the focal surfaces). The twelfth chapter deals with so-called congruences of Vasil'ev (here the second order surfaces of Lie adjoined to the focal surfaces at the foci of one ray are tangent to each other along the generators). The last chapter discusses the projective bending of surfaces and the conformal transformation of Tikhotzky [C. R. Acad. Sci. Paris **202**, 1474–1475 (1936)]. A historical survey and a bibliography conclude the book. [In this bibliography we miss Eisenhart's book, Transformations of Surfaces, Princeton Univ. Press, 1923]. *D. J. Struik.*

Finikov, S. P. On stratifiable pairs of congruences associated with an isotropic congruence. Doklady Akad. Nauk SSSR (N.S.) **73**, 899–900 (1950). (Russian)

This short note, which refers to the author's book, reviewed above, is related to the author's paper [Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 287–314 (1943); these Rev. **6**, 19] and the papers of S. Rossinski [C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 5–9, 54–56, 101–103 (1943); these Rev. **6**, 104], to the reviews of which we must refer for further elucidation, also with respect to the notation and the frame of reference used. The results of these papers are here complemented by the theorem that every stratifiable pair (with the possible exception of the case that $\alpha_1-\alpha_3=\frac{1}{2}\pi$) associated with an isotropic congruence of common perpendiculars is contained in a congruence of Bachvaloff which in this case must be an isotropic congruence [Rec. Math. [Mat. Sbornik] N.S. **6(48)**, 67–76 (1939); these Rev. **1**, 270].

D. J. Struik (Cambridge, Mass.).

Backes, Fernand. Sur un nouveau couple de surfaces projectivement applicables. C. R. Acad. Sci. Paris **232**, 33–34 (1951).

This paper states some results based on the author's previous paper [same C. R. **228**, 49–51 (1949); these Rev. **10**, 400]. A typical result may be stated as follows: Let two surfaces S_1 and S_2 be in one-to-one point correspondence and have the following properties. (1) Their asymptotic curves correspond. (2) They be in the relation of a transformation F. (3) The congruence of lines joining corresponding points be a W congruence, the asymptotic curves on the focal surfaces F_1, F_2 of the congruence corresponding to those on S_1, S_2. Under those conditions the congruence of lines of intersections of the tangent planes to S_1 and S_2 at corresponding points is a W congruence, the asymptotic curves on whose focal surfaces correspond to those on S_1 and S_2. If, moreover, the cross ratio of the corresponding points of S_1S_2 and the focal points on the line joining these points is constant, then S_1 and S_2 are projectively applicable in the sense of Čech. *V. G. Grove.*

Delgleize, A. Sur les transformations de Ribaucour et les quadriques. Bull. Soc. Roy. Sci. Liège **19**, 222–238 (1950).

This paper associates with minimal surfaces and surfaces of constant curvature related by a transformation R of Ribaucour certain quadrics of revolution. Let S and S' be the two surfaces related by the transformation R, and S_0 the surface of centers. Let these surfaces be generated by the respective points M, M', M_0. The plane of these points is called the plane of the transformation R. It is normal to the tangent plane to S_0 at M_0. Let S and S' be spherical surfaces, and let F_1, F_2 and F_1', F_2' be the unique surfaces parallel respectively to S and S' and having constant mean curvature. The lines F_1F_2', $F_1'F_2$ intersect in a point N whose locus Σ is the envelope of the plane of the transformation R. The surface Σ is isometric to a surface of revolution. Finally let S and S' have constant curvature a^{-1} and be related by a transformation R. Each of the points of one of the surfaces is located on the transform by reciprocal radii with respect to a quadric of revolution uniquely related to each point of the other surface. *V. G. Grove* (East Lansing, Mich.).

Delgleize, A. L'inversion et les transformations de Ribaucour. Bull. Soc. Roy. Sci. Liège **19**, 327–334 (1950).

The purpose of this paper is to prove the following theorem. Let S_1 and S_2 be R transforms of each other. With pole P_1 of S_1 and power -2φ invert the points P_2 of S_2 into points P of a surface Σ. Then Σ is an R transform of S_2, and Σ and S_1 are related by a transformation of Combescure.

V. G. Grove (East Lansing, Mich.).

Rozet, O. Sur certaines suites de Laplace de période six. Bull. Soc. Roy. Sci. Liège **19**, 243–249 (1950).

Rozet, O. Sur les suites de Laplace et les congruences de droites. Bull. Soc. Roy. Sci. Liège **19**, 335–342 (1950).

In the first of these papers the conditions that a sequence of Laplace, $\cdots, y_3, y_2, y_1, y, z, z_1, z_2, z_3, \cdots$, in a space of three dimensions be of period six are developed. In the second a study is made of the congruence of lines (y_1z_1). It is shown that if the developables of this congruence G correspond to the curves of the given conjugate net generating the sequence then the same is true for the congruence r_1s_1 of the sequence $\cdots, r_3, r_2, r_1, r, s, s_1, s_2, s_3, \cdots$, r and s being the focal points of G. Hence a type of transformation of congruences is found which preserves developables, that is,

the developables of the congruences of the sequence correspond. *V. G. Grove* (East Lansing, Mich.).

Marcus, F. Sur une propriété qui caractérise les surfaces de Jonas. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 906–910 (1950).

The defining differential equations of a surface S may be written in the form $x_{uu}=\theta_u x_u+\beta x_v+px$, $x_{vv}=\gamma x_u+\theta_v x_v+qx$. Then S is a surface of Jonas if $\beta_u=\gamma_v$. It is the purpose of this note to characterize geometrically this analytic condition. A congruence of Waelsh is one for which the asymptotic net on one focal surface corresponds to a conjugate net on the other. Only the surfaces of Jonas may serve as a focal surface for two congruences of Waelsh generated by the tangents to the curves of a conjugate net. *V. G. Grove.*

Saban, Giacomo. Sulle congruenze di Guichard. Boll. Un. Mat. Ital. (3) **6**, 3–8 (1951).

Using the notation of dual vectors, the author considers line congruences, whose (dual) line element can be brought into Tchebychef form. He develops a set of equations (of Frenet-Gauss-Weingarten type), and obtains a proof of the fact that any such congruence is a Guichard congruence. The integrability conditions imply then easily two equations which express known metric properties of Guichard congruences. *H. Samelson* (Ann Arbor, Mich.).

Mishra, R. S. Ratio of the distance to the central point and the parameter of distribution in a line congruence. J. Indian Math. Soc. (N.S.) **14**, 171–173 (1950).

Etant données une congruence rectiligne C et l'une quelconque R de ses surfaces réglées, l'auteur donne, au moyen du quotient de deux formes différentielles quadratiques, le rapport p de l'abscisse du point central situé sur un rayon quelconque (u_1, u_2) de R (compté à partir de la surface de référence de C) au paramètre de distribution de R relatif au même rayon (u_1, u_2). Lorsque du_1/du_2 varie, p admet un maximum et un minimum p_1 et p_2 (rapports principaux), et p_1+p_2, p_1p_2 sont dits respectivement rapports moyen et total. Il est montré que si le rapport moyen est nul, ou bien C est normale, ou bien la surface de référence est la surface moyenne. En ce qui concerne les extremums de p on a les résultats suivants: Si la surface de référence de C est la surface moyenne, si f et θ sont les deux formes de Sannia de C, et si $\mathcal{J}$ désigne l'opération jacobienne, les surfaces pour lesquelles p est extrémum sont les surfaces (caractéristiques) définies par l'équation $\mathcal{J}(\theta, \mathcal{J}(f, \theta))=0$. Si, dans l'équation précédente, on remplace θ par φ (2^{2me} forme fondamentale de Kummer), on obtient l'équation définissant, dans une congruence normale, les surfaces réglées pour lesquelles p est extrémum. *P. Vincensini* (Marseille).

Calapso, R. Invarianti di una superficie rispetto ad una trasformazione asintotica. Matematiche, Catania **5**, 98–102 (1950).

In an earlier note [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **25**, 105–106 (1937)] the author gave geometric characterizations of certain systems of lines on a surface which are invariant under asymptotic transformations. The present note is devoted to establishing analytically four asymptotic invariants. These in turn can be used to demonstrate the invariance of the systems of the cited note.

J. L. Vanderslice (College Park, Md.).

Buchdahl, H. A. Über die Variationsableitung von Fundamentalinvarianten beliebig hoher Ordnung. Acta Math. **85**, 63–72 (1951).

A fundamental invariant K of a Riemannian space-time, or more generally of any Riemannian space, is by definition a scalar function of the fundamental tensor $g_{\mu\nu}$ and its derivatives up to any given order. The variational (i.e. Hamiltonian) derivative $P^{\mu\nu}$ of K is defined by the variational equation

$$\delta\int K\sqrt{(-g)}d\tau=\int P^{\mu\nu}\delta g_{\mu\nu}\sqrt{(-g)}d\tau.$$

Formulae for the Hamiltonian derivatives of invariants involving no more than second-order derivatives of $g_{\mu\nu}$ are already known [Eddington, The Mathematical Theory of Relativity, Cambridge University Press, 1923, § 61; Buchdahl, Quart. J. Math., Oxford Ser. (1) **19**, 150–159 (1948); these Rev. **10**, 212]. In this paper the author obtains an explicit formula for the Hamiltonian derivative of the most general fundamental invariant in a form immediately applicable to special cases, a few of which he discusses at the end of the paper. He also gives a proof by direct calculation of the known theorem that the divergence of $P^{\mu\nu}$ is zero, and considers, too, the case when $K\sqrt{(-g)}$ is a gauge-invariant scalar-density. *H. S. Ruse* (Leeds).

Bodiou, Georges. Opérateur hermitien, de caractère stochastique, et opérateur covariant, de dérivation des spineurs, dans un espace de Riemann (à quatre dimensions). C. R. Acad. Sci. Paris **232**, 806–808 (1951).

The author discusses briefly and in general terms the statistical interpretation of operators on spinor fields defined over a Riemannian space. *A. H. Taub* (Urbana, Ill.).

Mogi, Isamu. On harmonic field in Riemannian manifold. Kōdai Math. Sem. Rep. **1950**, 61–66 (1950).

The author makes some interesting contributions (in part anticipated by Lichnerowicz) to the reviewer's theory of "curvature and Betti numbers" whose simplest result is that if the Ricci tensor on a compact Riemannian space is non-negative then the first Betti number vanishes. The author generalizes results by weakening somewhat the assumptions, and he also has some results involving homology and homotopy numbers without reference to positiveness or negativeness of the curvature tensor. For instance, he has some statements not only on the Betti number B_p, which is a topological invariant, but also on the number B_p' which is the maximal number of skew p-tensors which are not only harmonic but have vanishing first covariant derivative and which probably are not topological invariants for all p.

S. Bochner (Princeton, N. J.).

Lichnerowicz, André. Sur les variétés riemanniennes admettant une forme à dérivée covariante nulle. C. R. Acad. Sci. Paris **232**, 677–679 (1951).

A continuation of the investigations of the author [same C. R. **231**, 1413–1415 (1950); **232**, 146–147 (1951); these Rev. **12**, 535, 536] and H. Guggenheimer [ibid., 470–472 (1951); these Rev. **12**, 535] into the properties of compact Riemannian manifolds on which there exists an exterior form with zero covariant derivative. *W. V. D. Hodge.*

Salini, U. Sulle trasformazioni puntuali fra due spazi ordinari in una coppia di punti ad jacobiano nullo di caratteristica uno, quando la superficie jacobiana presenta un punto doppio uniplanare. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **9**, 26–53 (1950).

Continuing his study of the one-to-one punctual transformations between two projective S_3's [same Rend. (5) **8**, 229–245 (1949); these Rev. **12**, 51], in the neighborhood of a couple O, O' of corresponding points, where the Jacobian of the transformation is zero, the author studies here the case in which the Jacobian surface has a uniplanar point. The study is extended up to the neighborhood of the 2d order of the couple O, O', and the existence of intrinsic projective frames in the bundles of lines through O, O' is investigated. But, as it happened in the case studied in the cited paper, the consideration of the neighborhood of the 3d order is needed in order to complete the determination of the projective frames. [Remark: At the beginning of the 11th line of the review of the cited paper one should read 3d order instead of 2d order.] Moreover, many geometrical objects, attached to the transformation are studied, and canonical formulas for it are given. *V. Dalla Volta.*

Sangermano, Cosimo. Sulle trasformazioni puntuali fra due spazi ordinari in una coppia a jacobiano nullo, di caratteristica zero. Rivista Mat. Univ. Parma **1**, 375–382 (1950).

Study of a correspondence between two three-dimensional projective spaces in the neighborhood of two corresponding points (O, O') in case the Jacobian has rank zero. The correspondence between the directions at O, O' is (in general): The neighborhoods of the 2d order of O, O' determine an intrinsic reference system in the two stars of directions. The neighborhoods of order 3 and 4 determine intrinsically reference systems in the two spaces and therefore a canonical form for the transformation. *E. Bompiani.*

Mineo, Massimo. Paragone metrico d'intorni superficiali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7**, 84–87 (1949).

Remarks are made on a paper by V. Dalla Volta [same Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 64–68 (1949); these Rev. **10**, 625] on the applicability of two surface caps (calotte superficiali) of Euclidean S_3, up to a certain order in the neighbourhood of their centers. The inconsistency of the author's remarks [he bases his statements upon the false assumption that any correspondence between two surfaces: $z=f(x, y)$, $z=g(x, y)$ may be represented by taking as corresponding points those with the same x, y] has been proved by Dalla Volta and acknowledged by the author [see the two following reviews]. *E. Bompiani.*

Dalla Volta, Vittorio. Sulla isometria di calotte superficiali nello spazio euclideo. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 311–319 (1950).

Mineo in two recent notes [see the preceding review and the second following review] apparently has cast doubt upon the validity of certain results obtained by the author [Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 211–227 (1949); these Rev. **11**, 540]. The burden of the present note is to demonstrate the inadequacy of the Mineo approach and reaffirm the validity of the results attacked. The analysis of the various questions at issue is given in considerable detail. The reviewer does not hazard taking sides. *J. L. Vanderslice* (College Park, Md.).

Mineo, Massimo. Paragone metrico d'intorni superficiali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 546–550 (1950).

This apparently is an attempt to clarify and terminate a controversy between Dalla Volta and the author [cf. the preceding review] over isometries of surface patches, a subject first broached by Bompiani [Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **7**, 274–294 (1948); these Rev. **10**, 209]. The present paper [not to be confused with the second preceding review] refinds the results of Bompiani and Dalla Volta. It then defends a previous different result which the author obtained using geodesic polar (singular) coordinates and which helped start the controversy. *J. L. Vanderslice* (College Park, Md.).

Mineo, Massimo. Sul confronto dal punto di vista metrico differenziale di due intorni superficiali. Atti Accad. Sci. Lett. Arti Palermo. Parte I. (4) **8**, 113–119 (1949).

This note constitutes an intermediate phase in the Mineo-Dalla Volta controversy. Compare the paper reviewed above which can be considered to supersede the present note. *J. L. Vanderslice* (College Park, Md.).

Süss, Wilhelm. Eichflächenprinzipien in der projektiven Flächentheorie. J. Reine Angew. Math. **188**, 100–111 (1950).

The author's aim is to set the foundations of a general projective theory of surfaces in a manner roughly corresponding to Berwald's treatment of the Euclidean and affine case [Jber. Deutsch. Math. Verein. **31**, 162–170 (1922)] but strongly employing the methods of relative differential geometry. Eisenhart's so-called fundamental transformation plays an important role. Subsumption of the Fubini-Čech theory is emphasized. *J. L. Vanderslice.*

Kanitani, Jōyō. On a method of plunging of R_2 with symmetric projective connexion into a projective space S_4. Mem. Coll. Sci. Univ. Kyoto Ser. A. **25**, 87–91 (1949).

In this paper the author shows that a two-dimensional space R_2 with a symmetric projective connection can be imbedded in a four-dimensional projective space S_4 in such a way that R_2 becomes a surface V_2 and that a net of curves arbitrarily given on R_2 becomes the net of the characteristic curves of V_2, the characteristic curves of V_2 being defined to be the curves on V_2 along which the family of any covariant hyperplane associated with each point A_0 of V_2 and passing through the tangent plane t of V_2 at A_0 has t for its characteristic at A_0. For another way of imbedding R_2 in S_4 see the author's previous paper [Jap. J. Math. **19**, no. 4, 395–403 (1948); these Rev. **12**, 360]. *C. C. Hsiung* (Cambridge, Mass.).

Rozenfel'd, B. A. Projective geometry as metric geometry. Trudy Sem. Vektor. Tenzor. Analizu. **8**, 328–354 (1950). (Russian)

The fundamental idea of this work lies in the fact that the principle of duality of projective geometry is applicable to the space of m-pairs [Mat. Sbornik N.S. **24**(66), 405–428 (1949); these Rev. **11**, 133]. Thus a projective space P_n becomes a space of $2n$ dimensions whose elements are 0-pairs, i.e., points and their dual hyperplanes, the advantage being in the fact that such a space is metric, two 0-pairs determining a range of 4 points or dually a pencil of 4 hyperplanes. However, the author does not pursue this synthetic point of view but develops the subject analytically. This is based on the fact that the fundamental group

of P_n, $\mathfrak{P}_r$ is a noncompact simple Lie group. To each such group corresponds a compact simple Lie group in which the elements are complex. This is the $\mathfrak{R}_n$ of motions in an n-dimensional complex unitary-elliptic space K_n in which there is a real metric defined by a Hermitian form. If the points of this space are given by homogeneous coordinates ξ^i $(i=0, 1, \cdots, n)$, then the distance between two points is given by (*) $\cos^2 \omega/k=\xi^i\eta^{\bar{\imath}}\cdot\xi^{\bar{\imath}}\eta^i/\xi^i\xi^{\bar{\imath}}\cdot\eta^i\eta^{\bar{\imath}}$, where k is the "radius" of the space and is either real or pure imaginary. From the algebra of $\mathfrak{R}_n$ one obtains the Clifford algebra $(1, e, e^2=1)$ and the corresponding group $\mathfrak{B}_n$. If the basis is changed to $e_1=\frac{1}{2}(1+e)$, $e_2=\frac{1}{2}(1-e)$, then $e_1^2=e_1$, $e_2^2=e_2$, $e_1e_2=0$; the group $\mathfrak{B}_n$ is then also the group of motions of an n-dimensional unitary-elliptic space B_n but based on Clifford numbers $(\alpha=ae_1+be_2; \bar{\alpha}=be_1+ae_2)$. If in B_n $\xi^i=x^ie_1+u_ie_2$ then the metric (*) (with $k=1$) becomes $\cos^2 \omega=u_{\bar{\jmath}}y^{\bar{\jmath}}\cdot v_jx^j/u_jx^j\cdot v_{\bar{\jmath}}y^{\bar{\jmath}}$, which is a projective invariant since it is the cross ratio of two points and two hyperplanes. From this the line element of K_n or B_n is obtained by considering $\cos^2 d\omega$ in the form $(d\omega)^2=2\{\xi^i\xi^{\bar{\imath}}d\xi^jd\xi^{\bar{\jmath}}-\xi^id\xi^{\bar{\imath}}\cdot\xi^{\bar{\jmath}}d\xi^j\}/(\xi^i\xi^{\bar{\imath}})^2$ which was previously obtained by Schouten and van Danzig. By writing the metric in the form $(d\omega)^2=a_{\alpha\beta}-d\sigma^\alpha d\bar{\sigma}^{\bar{\beta}}$ where $a_{\alpha\bar{\beta}}$ is a Hermitian symmetric tensor one shows that $a_{\alpha\bar{\beta}}=\partial_\alpha\partial_{\bar{\beta}}\ln(\xi^i\xi^{\bar{\imath}})$. Thus the space P_n is stratified and the tensor analysis and geometry of such spaces has been previously developed by P. K. Raševskii and the author [the author, Mat. Sbornik N.S. 24(66) 53–74 (1949); these Rev. 11, 55]. *M. S. Knebelman* (Pullman, Wash.).

Ščerbakov, R. N. The repère of a line on a surface in affine differential geometry. Doklady Akad. Nauk SSSR (N.S.) 76, 655–657 (1951). (Russian)

The derivatives of the vectors of the canonical affine trihedral of a curve on a surface are linear forms in these vectors whose coefficients $\alpha, \beta, \gamma, \mu, \pi$ are curve invariants. For a curve on a surface these invariants are expressible in terms of the arc of the curve and surface invariants. Thus α, β and π are analogues of the normal curvature, geodesic curvature and geodesic torsion. The author considers a number of special problems of curves whose invariants satisfy various relations. Thus lines of Darboux are characterized by $\alpha=0$; lines of Segre by $\beta-\gamma=0$; affine geodesic lines by $\beta+\gamma$, etc. *M. S. Knebelman* (Pullman, Wash.).

Winogradzki, Judith. Sur la connexion des espaces affines. C. R. Acad. Sci. Paris 232, 936–938 (1951).

The author defines a generalization of parallel displacement of tensors. The effect of successively imposing the conditions of commutivity of this generalized parallel displacement with (i) addition, (ii) addition and multiplication, and (iii) addition, multiplication, and contraction is studied. In the last case, the displacement reduces to the classical parallel displacement. *A. Fialkow* (Brooklyn, N. Y.).

Levine, Jack. Collineations in Weyl spaces of two dimensions. Proc. Amer. Math. Soc. 2, 264–269 (1951).

In a previous paper [Ann. of Math. (2) 52, 465–477 (1950); these Rev. 12, 130], the author studied the collineations in projectively and affinely connected spaces of two dimensions. He proved there that there are 12 types of affinely connected spaces of two dimensions admitting affine collineations (that is to say, point transformations which change each path of the space into a path and preserve the affine parameters on paths). In the paper under review, the author tested each of these types for reducibility to a Weyl space (which is characterized by $g_{ij,k}+2g_{ij}\phi_k=0$) of two dimensions and showed that there will remain only three types admitting respectively affine collineations of one, two and three parameters. *K. Yano* (Princeton, N. J.).

Raševskiĭ, P. K. On a pair of connections on n-dimensional surfaces in a $2n$-dimensional stratified space. Trudy Sem. Vektor. Tenzor. Analizu. 8, 301–313 (1950). (Russian)

The space under consideration is one of $2n$ dimensions in which there exists a coordinate system $x^1, \cdots, x^n, u^{\bar{1}}, \cdots, u^{\bar{n}}$ and a metric $ds^2=\{\partial^2U(x, u)/\partial x^i\partial u^{\bar{\jmath}}\}dx^idu^{\bar{\jmath}}$. This metric is studied under transformations among the x's and among the u's separately. If S_n is a subspace, $x^i=x^i(t^1, \cdots, t^n)$, $u^{\bar{\imath}}=u^{\bar{\imath}}(t^1, \cdots, t^n)$, the two Jacobians being different from zero, then a vector ξ^α in S_n has components in the enveloping space $\xi^i=(\partial x^i/\partial t^\alpha)\xi^\alpha$, $\xi^{\bar{\imath}}=(\partial u^{\bar{\imath}}/\partial t^\alpha)\xi^\alpha$ and since magnitude is to be preserved, the induced metric in S_n is given by $G_{\alpha\beta}=(\partial x^i/\partial t^\alpha)(\partial u^{\bar{\imath}}/\partial t^\beta)g_{i\bar{\jmath}}$, $g_{i\bar{\jmath}}=\partial^2U/\partial x^i\partial u^{\bar{\jmath}}$. Parallel displacement may be defined in two ways: with respect to the space $u=$const. or $x=$const. which yields two affine connections $\Lambda^\alpha{}_{\beta\gamma}$ and $M^\alpha{}_{\beta\gamma}$ and the requirement that length is to be preserved under parallel displacement gives

$$(*) \qquad \partial G_{\mu\lambda}/\partial t^\nu-\Lambda^\alpha{}_{\mu\nu}G_{\alpha\lambda}-M^\alpha{}_{\lambda\nu}G_{\mu\alpha}=0.$$

The author then considers a projectively flat space S_n with two connections and a semimetric tensor $G_{\mu\lambda}$ satisfying (*) and shows that if $\Lambda^\alpha{}_{\beta\gamma}=\delta_\beta{}^\alpha\Lambda_\gamma+\delta_\gamma{}^\alpha\Lambda_\beta$ and $\Lambda_{\gamma\beta}=\partial\Lambda_\gamma/\partial t^\beta-\Lambda_\beta\Lambda_\gamma$ then $\Lambda_{\gamma\delta}=M_{\delta\gamma}=\Lambda G_{\gamma\delta}$ where Λ is necessarily a constant. The author then constructs a simple realization of a stratifiable space in which x^i are the nonhomogeneous coordinates of a point in P_n and $u^{\bar{\imath}}$ the nonhomogeneous coordinates of its dual hyperplane. If incidence is defined by $x^iu^{\bar{\imath}}+1=0$ and $U=\ln(x^iu^{\bar{\imath}}+1)$, one obtains the projectively flat connection in which $\Lambda_{ij}=-u^{\bar{\imath}}/\{x^iu^{\bar{\imath}}+1\}$. Conversely every projectively flat connection for which $|\Lambda_{ij}|\neq 0$ may be realized as a subspace S_n of a stratifiable space of $2n$ dimensions.

M. S. Knebelman (Pullman, Wash.).

Džavadov, M. On a realization of a stratified space. Trudy Sem. Vektor. Tenzor. Analizu. 8, 314–327 (1950). (Russian)

An example of a 4-dimensional stratified space is obtained by considering a 3-dimensional (Euclidean space) with Cartesian coordinates (x, y, z) and a one-parameter group G of mappings $x^1=xe^t$. A family of planes (u, v, w) whose equation is $ux+vy+wz-1=0$ is mapped into the planes $ue^{-t}x+ve^{-t}y+we^{-t}z-1=0$. If one considers the points for which $z=1$ and the planes for which $w=1$, one gets an element (x, y, u, v) for which t may be taken as the metric function [see the preceding review]. Taking the $U=\ln(x^1u^1+x^2u^2+1)$, the author works out the components $\Gamma^i{}_{jk}$ and g_{ij} as well as the curvature tensor and the equations of the geodesics in finite form.

M. S. Knebelman (Pullman, Wash.).

Su, Buchin. Lie derivation in the geometry of conformal connections. Acad. Sinica Science Record 2, 331–339 (1949).

The author considers the problem of finding whether a space with a normal conformal connection admits infinitesimal transformations. The conditions of integrability are written in a simple form by means of a generalization of the operation of Lie derivation. *E. T. Davies.*

Su, Buchin. Extremal deviation in a geometry based on the notion of area. Acta Math. **85**, 99–116 (1951).

This paper is concerned with Cartan spaces, which are based upon the notion of area as the fundamental metrical notion. Cartan [Les espaces métriques fondés sur la notion d'aire, Actualités Sci. Ind., no. 72, Hermann, Paris, 1933] has defined hypersurfaces and extremal hypersurfaces in such spaces. In this paper the author considers the deformation of an extremal hypersurface into a nearby extremal hypersurface, and finds the changes in the tensors characterizing the hypersurface due to the deformation. He shows how these changes depend upon the curvature and the torsion of the space. At the end of the paper he gives an indication of a generalization to the case where the deforming function depends both upon the hypersurface element and on the position. *E. T. Davies* (Southampton).

Su, Buchin. Axiom of the plane in a space of K-spreads. Acad. Sinica Science Record **3**, 7–16 (1950). (English. Chinese summary)

A subspace S of a space of K-spreads is called flat at a point P if S contains all the K-spreads which pass through P. If S is flat at every point it is said to be totally flat. In this terminology the axiom of the plane is stated as follows: For a certain L $(1<K<L<N)$, through each point and tangent to each L-vector at that point there passes a twice differentiable totally flat subspace of L dimensions. In this paper it is proved that if the axiom of the plane holds in a space, then the space is descriptively flat. *E. T. Davies.*

Ku, Chao-Hao. New treatment of geometries in a space of K-spreads. Acad. Sinica Science Record **3**, 41–51 (1950). (English. Chinese summary)

In this paper the author uses the implicit function form for the equations of K-spreads instead of the parametric form generally used. He obtains invariants corresponding to those obtained by J. Douglas [Math. Ann. **105**, 707–733 (1931)] and defines three classes of geometries of K-spreads. *E. T. Davies* (Southampton).

Ku, Chao-Hao. On the descriptive geometry of a space of K-spreads. Acad. Sinica Science Record **3**, 53–59 (1950). (English. Chinese summary)

The author gives relations between his implicit function method of treating K-spreads, and the parametric method of Douglas. He also gives some theorems about the intersection of K-spreads. *E. T. Davies* (Southampton).

Gołąb, St. Sur les objets géométriques à une composante. Ann. Soc. Polon. Math. **23**, 79–89 (1950).

This paper deals with geometric objects of the first class with one component. In a preceding paper [same Ann. **20**, 10–27 (1948); these Rev. **10**, 267] the author has shown that these geometric objects in an X_n $(n>2)$ are of the Δ-type. For an X_2, however, these geometric objects are of two kinds, which are represented by (1) a density (objects of the Δ-type) and (2) the quotient of the components of a vector. *J. Haantjes* (Leiden).

Gołąb, S. La notion de similitude parmi les objets géométriques. Bull. Int. Acad. Polon. Sci. Lett. Cl. Sci. Math. Nat. Sér. A. Sci. Math. **1950**, 1–7 (1950).

A geometric object Ω^* is similar to another one Ω if its components Ω_j^*, can be written as functions of the components of Ω: $\Omega_j^*=g_j(\Omega_i)$ and if the transformation $\Omega\to\Omega^*$ is invertible. It is proved that if Ω is a special geometric object (classification of Schouten and Haantjes [Proc. London Math. Soc. (2) **42**, 356–376 (1937)]), Ω^* is also a geometric object. If Ω is a Weyl density and the transformation is neither invertible nor even, Ω^* need not be a special geometric object. In three interesting remarks at the end of the paper the author shows that the definition of similarity is in fact reasonable. *J. A. Schouten* (Epe).

Tachibana, Syun-ichi. On Finsler spaces which admit a concurrent vector field. Tensor N.S. **1**, 1–5 (1950).

It is shown that an n-dimensional Finsler space contains a concurrent vector field if and only if the line element can be written in the form

$$ds^2=(dx_n)^2+(x_n)^2H(x_1,\cdots,x_{n-1},dx_1,\cdots,dx_{n-1}),$$

where H is the space of the line element of some $(n-1)$-dimensional Finsler space. In more geometric language it may be said that a Finsler space with a concurrent vector field contains a congruence of extremals and family of hypersurfaces which have everywhere umbilics (in an appropriate sense) and are transversal to the congruence. *H. Busemann* (Los Angeles, Calif.).

Ehresmann, Charles, and Libermann, Paulette. Sur les structures presque hermitiennes isotropes. C. R. Acad. Sci. Paris **232**, 1281–1283 (1951).

For an almost-Hermitian structure in a manifold V^{2n} defined by n independent complex-valued forms ω_i and the Hermitian form $\sum\omega_i\bar{\omega}_i$, the structure equations, expressing $d\omega_i$ in terms of ω_j and $\bar{\omega}_j$ are set up, and from them a first and second torsion and a curvature defined. A proof is indicated for theorem 1 that an isotropic structure is locally homogeneous and locally equivalent to either elliptic or hyperbolic or flat Hermitian geometry; isotropic at a point x means the existence of sufficiently many automorphisms of the structure with x fixed, locally homogeneous means the existence of automorphisms of the structure sending x into any nearby x'. For restricted isotropy, where one insists on unimodular unitary transformations, one obtains (theorem 2) either the same as in theorem 1 or (locally) a specific structure on S^6, for which G_2 appears as group of automorphisms. This last structure is not true-complex, since its second torsion does not vanish; it is the only possible locally homogeneous structure on S^6. *H. Samelson.*

Libermann, Paulette. Problèmes d'équivalence relatifs à une structure presque complexe sur une variété à quatre dimensions. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 742–755 (1950).

An almost complex structure in a manifold V^{2n} is given by n independent complex-valued differential forms ω_i. The structure equations read $d\omega_j=\omega_i\wedge\omega_{ij}+\Gamma_j$ where the Γ_j, the torsion forms, contain all the terms of the type $\bar{\omega}_r\wedge\bar{\omega}_s$. Vanishing of the torsion is characteristic for a true complex structure. The author treats, for $n=2$, the problem of equivalence of structures with torsion. Following Cartan, the principle is to make use of the admissible transformation of the ω_i by means of complex substitutions to arrive at normal forms for the structure equations with as many coefficients made constant as possible. The normal forms are arranged in a number of cases, depending e.g. on whether certain forms are completely integrable. Some of the structures admit infinite (in Cartan's sense) groups of automorphisms. The corresponding problem is also solved for an almost Hermitian structure, where a Hermitian differential

form is given. The transformations are then restricted to the unitary group. Two torsions, and a curvature are defined, in analogy to Hermitian geometry, and normal forms established. In one of the case a 5-parameter Lie group of automorphisms appears. *H. Samelson* (Ann Arbor, Mich.).

Bochner, S. Tensor fields with finite bases. Ann. of Math. (2) **53**, 400–411 (1951).

The paper gives some general finite basis theorems on the module of tensors or tensor densities of any mixed type on a compact manifold, which are harmonic in a certain general manner. The latter is a generalization of the harmonic property of a skew-symmetric tensor and is expressed by the condition that the components of the tensor or tensor density field satisfy a certain system of partial differential equations of the second order. It is also proved that there are finite sets of points on the manifold such that any element of the module vanishing on them vanishes identically. A particular case is the module of all holomorphic tensor densities on a compact complex manifold. Similar theorems are given concerning what the author calls the conglomerate and saltus functions on a compact complex manifold. Furthermore, singularities are also admitted. The results are further generalized on the universal covering space for finite groupings of the tensors which reproduce themselves by arbitrary bounded representations of the fundamental group. *S. Chern* (Chicago, Ill.).

Schrödinger, Erwin. On the differential identities of an affinity. Proc. Roy. Irish Acad. Sect. A. **54**, 79–85 (1951).

An arbitrary scalar density which is a function of the coefficients of an affine connection and their derivatives is considered as a Lagrangean function. The integral of this Lagrangean must be invariant under variations in the affine connection produced by coordinate transformations. From this fact four differential identities are deduced.

A. H. Taub (Urbana, Ill.)

Scherk, Peter, and Kwizak, Michael. What are tensors? Amer. Math. Monthly **58**, 297–305 (1951).

NUMERICAL AND GRAPHICAL METHODS

Gupta, Hansraj. Tables of distributions. Res. Bull. East Panjab Univ. **1950**, 13–44 (1950).

Let $u(n, a)$ denote the number of ways that n distinct objects may be assigned to a boxes. Then

$$\sum_{n=0}^{\infty} u(n, a)x^n/n! = (e^x-1)^a/a!.$$

The sum function $U(n)=\sum_{a=1}^{n}u(n, a)$ is generated by $\sum_{n=0}^{\infty}U(n)x^n/n=e^{-1}\exp(e^x)$. Tables are given of $u(n, a)$ for $1\leqq a\leqq n\leqq 50$ and of $U(n)$ for $n=1(1)50$. Some asymptotic results concerning $U(n)$ and the value of a for which $u(n, a)$ is a maximum are discussed. *D. H. Lehmer.*

***Todd, John. Table of Arctangents of Rational Numbers.** National Bureau of Standards, Applied Mathematics Series, No. 11. United States Government Printing Office, Washington, D. C., 1951. xi+105 pp. $1.50.

There are many publications in which $\tan^{-1}x$ is needed for certain rational values of x, in fact, for $x=m/n$, where m and n are moderately small integers. For example $\ln(x+iy)$ with x and y rational—they can be so chosen in many calculations—involves $\tan^{-1}y/x$, also rational. Such values can be computed by interpolation in existing tables, for example, the NBS Table of arc tan x[Math. Tables no. 16, National Bureau of Standards, 1942; these Rev. **4**, 89]. For $m, n<100$, however, values are needed with sufficient frequency to make permanent record desirable.

The present tables give $\tan^{-1}m/n$ and $\cot^{-1}m/n$ for all integers $m<n\leqq 100$ to 12 decimals in table 1, arranged in ascending order of m for each n in turn, the n's also being in ascending order. It is well known that $\tan^{-1}m/n$ can be expressed as an algebraic sum of several values of $\pm\tan^{-1}N$ for appropriate integer values N; also that linear relations exist connecting the $\tan^{-1}N$ for certain sets of values of N. For example,

$$\tan^{-1}(15/19)=6\tan^{-1}1-\tan^{-1}2-\tan^{-1}5-\tan^{-1}138.$$

As an example of the second kind

$$\tan^{-1}2+\tan^{-1}3=3\tan^{-1}1=0$$

whereby $\tan^{-1}3$ can be expressed in terms of $\tan^{-1}2$ and $\tan^{-1}1$, and $\tan^{-1}3$ (or 3 itself) is called reducible. In virtue of these relations only about 70% of the values for integer N are linearly independent and need special calculation. Table 2 gives a list of reducible integers N ($\leqq 2089$) and also the reductions of $\tan^{-1}N$ in terms of values of $\tan^{-1}N'$, all values of N' in each reduction being less than N. In table 1, the reduction of $\tan^{-1}m/n$, in terms of values of $\tan^{-1}N'$, is given; values of N' are here less than $\frac{1}{2}(m^2+n^2)$. In each case N' denotes an irreducible integer. An interesting introduction describes computational processes used in the construction of the tables, and the theory behind them.

J. C. P. Miller (Cambridge, England).

Inman, S. The probability of a given error being exceeded in approximate computation. Math. Gaz. **34**, 99–113 (1950).

The author gives elementary derivations of the error distributions of simple arithmetical functions of elements a_i affected by rectangularly distributed errors e_i. The functions of the a_i considered are simple operations such as sums, products, ratios, etc., and the resulting error distributions are sometimes approximated by a normal distribution. It is stressed that the concept of "proportionate error" is more precise than that of "number of significant figures."

H. O. Hartley (London).

Nobile, Amedeo. Calcoli con numeri approssimati. Archimede **3**, 62–69 (1951).

Lévy, Paul. Sur l'emploi des méthodes d'interpolation dans les mathématiques appliquées. Enseignement Math. **39** (1942–1950), 22–33 (1951).

Rosser, J. Barkley. Transformations to speed the convergence of series. J. Research Nat. Bur. Standards **46**, 56–64 (1951).

This paper justifies certain particular cases of the application of Euler's transformation

$$\sum_{n=0}^{\infty}a_nx^n=\frac{1}{1-x}\sum_{n=0}^{\infty}(\Delta^n a_0)\left(\frac{x}{1-x}\right)^n$$

useful in the practical summation of series. More precisely,

the author shows rigorously that it may be used, among other series, for the series involved in the asymptotic expansion for the exponential integral of negative argument, namely $1-1/x+2!/x^2-3!/x^3+\cdots$ and for the similar error integral series $1-1/2w^2+1\cdot3/(2w^2)^2-1\cdot3\cdot5/(2w^2)^3+\cdots$. He then goes on to show how the convenient converging factor of Airey [Philos. Mag. (7) **24**, 521–552 (1937)] may be arrived at by a process less empirical than Airey's, thus justifying use of this factor. Next, he sounds a note of warning by applying the Euler transformation to the Euler-Maclaurin expansion for $\int_0^1 dx/(1+x^2)$, namely, $\frac{1}{2}-\sum_{m=0}^{n}\{(-1)^m B_m/m2^{m+1}\}\sin\frac{1}{2}\pi m$ in which case the result is clearly different from $\frac{1}{4}\pi$. This indicates the need forcare, but not, as the author suggests, that the Euler transformation is in this case inapplicable. The discrepancy here arises from a too complete identification of the series with the integral $\int_0^1 dx/(1+x^2)$. Several summation methods can be applied to the series, producing a consistent sum; to this sum the integral is an approximation which can be improved, either by means of a different integral, or by means of a series of which $\int_0^1 dx/(1+x^2)$ is the first and predominant term. Next the method of Bickley and Miller is outlined [see Philos. Mag. (7) **22**, 754–767 (1936)] as applied to $\sum(1/n^2)$, and finally a method involving transformation of a series into a continued fraction, described as being the most widely applicable method known to the author. This is given by Wall [Analytic Theory of Continued Fractions, Van Nostrand, New York, 1948; these Rev. **10**, 32]. Numerical applications are given throughout.

J. C. P. Miller (London).

Meulenbeld, B. A graphical method for determining the roots of a numerical high-degree equation with real coefficients. Simon Stevin **28**, 60–80 (1951). (Dutch)

The author makes a detailed study of a graphical method of finding the roots of (1) $f(z)=a_0z^n+a_1z^{n-1}+\cdots+a_n=0$, with real coefficients, due to E. Lill [Nouvelles Ann. Math. (2) **6**, 359–362 (1867); **7**, 363–367 (1868)]. Each polynomial is represented by an orthogon (Dutch "orthogoon"), a sequence of line segments $P_0P_1P_2\cdots P_{n+1}$ where the sides P_jP_{j+1} are proportional to $|a_j|$ and $P_jP_{j+1}P_{j+2}$ is a right angle turning to the left or right according as $a_ja_{j+1}\gtrless0$. The determination of a real root depends on finding a second orthogon with n sides whose interior vertices lie on the lines $P_1P_2, \cdots, P_{n-1}P_n$ and whose end-points are also P_0 and P_n. Many properties of the real roots of a polynomial are proved geometrically by means of these orthogons; for example, (i) the condition that a cubic have a double root, and (ii) the condition that a pair of quadratics have a common root, are derived. Geometric constructions are also given for the values of $f(z), f'(z), \cdots, f^{(n)}(z)$, and $(f(z)-f(x))/(z-x)$ when z and x are real. An extension to the complex roots of (1) first indicated by Lill, is developed more fully.

A. W. Goodman (Lexington, Ky.).

Falk, Sigurd. Ein übersichtliches Schema für die Matrizenmultiplikation. Z. Angew. Math. Mech. **31**, 152–153 (1951).

Vogel, Alfred. Zur Bestimmung der Eigenwerte einer Matrix durch Iteration. Z. Angew. Math. Mech. **30**, 174–182 (1950). (German. English, French and Russian summaries)

The practical evaluation of characteristic values and characteristic vectors of matrices is discussed and illustrated by numerical examples. In particular, certain well-known iterative methods for the determination of the maximum characteristic value and the second largest characteristic value (as well as their corresponding characteristic vectors) are discussed in some detail. It is proposed to calculate the smallest characteristic values of a matrix once a scheme for the determination of the highest characteristic values is known, by the following artifice. Given, $\lambda_1>\lambda_2>\cdots>\lambda_n$ are real characteristic values of a matrix U. Set up the related matrix $R=\lambda_1 I-U$. Then by a theorem due to Frobenius the characteristic values of R are given by $\rho_1=0$, $\rho_2=\lambda_1-\lambda_2, \cdots, \rho_n=\lambda_1-\lambda_n$, where $\rho_1<\rho_2<\cdots<\rho_n$.

H. Polachek (White Oak, Md.).

Vogel, Alfred. Zur Berechnung der Torsionseigenschwingungen von Maschinenwellen. Z. Angew. Math. Mech. **30**, 363–369 (1950). (German. English, French, and Russian summaries)

The author sets up a matrix whose charcteristic roots are the squares of the natural frequencies of a given oscillating system. He reduces the problem to a discussion of the characteristic roots of a real symmetric matrix $Z=(z_{ij})$ whose elements z_{ij} and $z_{i,i+1}=z_{i+1,i}$ are positive while $z_{ij}=0$ when $j>i+1$ $(i, j=1, \cdots, n)$. The roots are given explicitly in the special case where $z_{ij}=2c$, $z_{i,i+1}=c$. In the general case, the author recommends iteration methods and refers to an earlier paper for treatment of a numerical case [see the preceding review].

G. B. Thomas.

O'Brien, George G., Hyman, Morton A., and Kaplan, Sidney. A study of the numerical solution of partial differential equations. J. Math. Physics **29**, 223–251 (1951).

The authors present in a systematic and unified manner, a discussion of methods developed in recent years for the analysis and improvement of the stability characteristics (error growth) of finite difference equations used in the numerical solution of systems of partial differential equations. There is also a discussion of the convergence of the solutions of finite difference equations to the solutions of the corresponding differential equations. A method of stability analysis, due primarily to J. von Neumann, is discussed in detail and is applied to several basic types of equations. The simplest form of the heat conduction equation (1) $\partial\phi/\partial t=\partial^2\phi/\partial x^2$, is used to illustrate the difficulties which may be encountered in carrying out the numerical solution of partial differential equations, unless a preliminary analysis is carried out. The following three equivalent difference forms are considered:

$$(2)\quad \frac{\phi(x,t+\Delta t)-\phi(x,t-\Delta t)}{2\Delta t}=\frac{\phi(x+\Delta x,t)-2\phi(x,t)+\phi(x-\Delta x,t)}{(\Delta x)^2},$$

$$(3)\quad \frac{\phi(x,t+\Delta t)-\phi(x,t)}{\Delta t}=\frac{\phi(x+\Delta x,t)-2\phi(x,t)+\phi(x-\Delta x,t)}{(\Delta x)^2},$$

$$(4)\quad \frac{\phi(x,t+\Delta t)-\phi(x,t)}{\Delta t}=\frac{1}{2(\Delta x)^2}\times\{[\phi(x+\Delta x,t+\Delta t)-2\phi(x,t+\Delta t)+\phi(x-\Delta x,t+\Delta t)]+[\phi(x+\Delta x,t)-2\phi(x,t)+\phi(x-\Delta x,t)]\}.$$

Form (2) was used by L. F. Richardson, and proves to be unstable for all values of $\Delta t/(\Delta x)^2$. Form (3) gives useful results only in case the inequality $\Delta t/(\Delta x)^2 \leqq \frac{1}{2}$ is satisfied. Form (4) is due to J. von Neumann and D. R. Hartree and gives stable results for all values of $\Delta t/(\Delta x)^2$. The discussions are supplemented by extensive numerical examples. The material should prove to be of considerable value in the practical solution of partial differential equations (by finite difference methods) on high speed digital calculators.

H. Polachek (White Oak, Md.).

Whittaker, E. J. W. Evaluation of Fourier transforms by a Fourier synthesis method. Acta Cryst. 1, 165–167 (1948).

The author is concerned with the numerical evaluation of the inverse $r\rho(r)$ of the Fourier transform

$$f(s) = K\int_0^\infty r\rho(r)\sin 2\pi srdr.$$

It is assumed that $r\rho(r)$ is negligible for $r \geqq r_a$ and expandible in a Fourier series (1) $r\rho(r) = \sum_{n=0}^{n} A_n \sin n\pi r/r_a$, $0 \leqq r \leqq r_a$, so that the coefficients A_n are expressible in the form $A_n = \text{const.} \times f(n/2r_a)$. The numerical evaluation of $r\rho(r)$ from (1) then requires an ordinary synthesis for which strips like the Beevers-Lipson strips [Proc. Phys. Soc. **48**, 772–780 (1936)] but at 3° interval and up to the 45th harmonic, have been prepared by the author. *H. O. Hartley.*

Cochran, W. A critical examination of the Beevers-Lipson method of Fourier series summation. Acta Cryst. 1, 54–56 (1948).

To facilitate the customary process of Fourier synthesis in crystallographic X-ray analysis Lipson and Beevers [Proc. Phys. Soc. **48**, 772–780 (1936)] prepared card strips giving to the nearest integer values of $C\cos(hm\,6°)$ and $C\sin(hm\,6°)$ for integral values of h, m, and C. For a particular combination of h and C a strip provides the above for a sequence of m-values, the answers being shown in standard positions running from left to right across the strip. A synthesis of the form $\sum_h C(h)\begin{matrix}\cos\\ \sin\end{matrix}(hm\,6°)$ can therefore be performed simultaneously for a sequence of m-values by selecting the strips corresponding to the required combinations of h and C, aligning them, and adding the answers down each m-column. The author deals with the computation of the electron density $\rho(x, y, z)$ in a three stage Fourier synthesis of the form (1) $\rho(x, y, z) =$

$$\sum_h \begin{matrix}\cos\\ \sin\end{matrix}\left(2\pi\frac{hx}{a}\right)\sum_k \begin{matrix}\cos\\ \sin\end{matrix}\left(2\pi\frac{ky}{b}\right)\sum_l \begin{matrix}\cos\\ \sin\end{matrix}\left(2\pi\frac{lz}{c}\right)F(h, k, l)$$

in which the quantities inside the { } are used in turn as multipliers C. The accuracy of ρ depends on (a) errors of measurement in the Fourier coefficients $F(h, k, l)$, (b) truncation of the series, and (c) accumulation of the rounding off errors in the computation of (1) with the help of the integral values on the Beevers-Lipson strips. Since the source of error (b) can be avoided [Bragg and West, Philos. Mag. (7) **10**, 823–841 (1930)] the author shows that with the present experimental technique source (a) is still responsible for a predominantly larger share of error than source (c). As a measure of error the author uses the expectation of the root mean square error or "standard deviation" attributable to each source. *H. O. Hartley* (London).

Rutishauser, Heinz, Speiser, Ambros, und Stiefel, Eduard. Programmgesteuerte digitale Rechengeräte (elektronische Rechenmaschinen). I, II, III, IV. Z. Angew. Math. Physik **1**, 277–297, 339–362 (1950); **2**, 1–25, 63–92 (1951) = Mitt. Inst. Angew. Math. Zürich **2**, 102 pp. (1951).

In this series of papers the authors discuss in very considerable detail a number of the important mathematical questions that naturally arise in the design of a digital computer. These topics include possible number systems, the questions of "fixed" vs "floating" point and complementation, the arithmetic processes, the grouping of numbers to achieve higher than normal precisions, conversion between number systems, the structure of finite approximation methods, error analysis, programming and coding as well as the physical organs of a machine. In many of these considerations the authors have compared the various points of view expressed by others in the field to give a comprehensive picture of the situation as understood at the present time. *H. H. Goldstine* (Princeton, N. J.).

Speiser, Ambros P. Entwurf eines elektronischen Rechengerätes unter besonderer Berücksichtigung der Erfordernis eines minimalen Materialaufwandes bei gegebener mathematischer Leistungsfähigkeit. Mitt. Inst. Angew. Math. Zürich **1**, ii+67 pp. (1950) = Thesis, Eidgenössiche Technische Hochschule in Zürich, 1950.

In this paper the author gives a quite complete account of an electronic digital computer, being considered by the computation laboratory at the Technical Institute in Zurich. The machine is to have a magnetic drum memory of 1200 word capacity, each word being 12 decimal digits. The multiplication speed is 30 ms. and the average access time to the memory is 16 ms. The author gives a comprehensive discussion of all relevant mathematical phases of the problem as he poses it. This is to construct a low cost, rapid machine for a modest size computation laboratory. The machine described is to have about 1000 vacuum tubes and 300 relays. *H. H. Goldstine* (Princeton, N. J.).

Mirimanoff, D. Description d'une famille d'appareils pour diviser un angle en un nombre quelconque de parties égales. Enseignement Math. **39** (1942–1950), 61–68 (1951).

Lindinger, Eckart. Eine Vorrichtung zur mechanischen Ausgleichung nach der Methode der kleinsten Quadrate. Z. Österreich. Ing. Architekt.-Verein. **96**, 74–76 (1951).

Hain, K. Die zeichnerische Bestimmung der Schleppkurven. Ing.-Arch. **18**, 302–309 (1950).

Tractrix curves are traced by the rear wheels of vehicles and the roller of a polar planimeter. They are constructed graphically by making use of the fact that the tractrix may be described more readily as an evolute by rolling the axis of the wheel over a curve. The latter curve is constructible from the path of the front wheel. The method is extended to determine the tractrix curves of wheels mounted on a linkage mechanism. *M. Goldberg* (Washington, D. C.).

Lah, Ivo. Eine praktische Interpolationsformel des Zinsfussproblemes von hoher Präzision. Mitt. Verein. Schweiz. Versich.-Math. **51**, 91–100 (1951).

Let $a_0 = a_x(i_0)$, $a_1 = a_x(i_1)$, $a = a_x(i)$ be the present values of the annuities for age x computed at the rate of interest i_0, i_1, i respectively. The author approximates a by

$\bar{a}=a^{(1)}(1-k_1)+a^{(2)}k_1$. Here $a^{(1)}$ (resp., $1/a^{(2)}$) is obtained by linear interpolation between a_0 and a_1 [resp., $1/a_0$ and $1/a_1$] while $k_1=k_1(x+1, i)=S^{(2)}_{x+1}N_{x+1}/(S_{x+1})^2$ is Poukka's ratio for age $x+1$ and interest i. *E. Lukacs.*

Sternhell, Charles M. Calculation of approximate annuity values on a mortality basis that provides for future improvements in mortality. Soc. Actuar. Trans. **2**, no. 3, 30–75 (1950).

It is supposed that the future rates of mortality q_x of each age x after n years will be at the level $q_x \cdot s_x{}^n$. Under these conditions the author works out formulae for approximate calculation of annuity values utilizing several special commutation columns. *P. Johansen* (Copenhagen).

Steffensen, J. F. More about invalidity functions. Skand. Aktuarietidskr. **33**, 193–202 (1950).

In case the mortality force of active persons is supposed to be a linear function of the mortality force of the mixed group of active and invalids, the author finds the simplest form of the different relations between the invalidity functions. A numerical example illustrates the method to find the constants of the linear function in question.

P. Johansen (Copenhagen).

ASTRONOMY

Agostinelli, Cataldo. Nuovi contributi alla teoria della figura dei pianeti. Univ. e Politecnico Torino. Rend. Sem. Mat. **9**, 179–224 (1950).

The classic discussions of the problem of the configurations of rotating fluid masses are generalized by the assumption of a complementary motion due to a dilatation. In this paper the latter condition is assumed for a heterogeneous fluid mass which is supposed to rotate in the gravitational field of far centers of attraction. On noting the results of the French investigators, as represented by Appell [Traité de mécanique rationnelle, tome 4, 2d ed., Gauthier-Villars, Paris, 1932, 1937], the author proceeds to the solution of the equation $U_i+\frac{1}{2}\omega^2(x^2+y^2)+C_1(x^2-z^2)+C_2(y^2-z^2)=\text{const.}$ by which the surfaces of equal densities are determined. The terms in C_i are due to the centers of attraction. The angular velocity ω is assumed to be small. Then, the potential U_i for a stratification differing but little from the ellipsoidal one is given in terms of the polar and equatorial ellipticities. It is shown that the equatorial ellipticity (which is due to the attraction of the far centers) satisfies an equation similar to the Clairaut equation. The values of the ellipticities are derived for the free surface of the fluid mass in the first approximation and, on assuming Roche's law for the distribution of density, the results are applied to the Earth. The equatorial ellipticity due to luni-solar attraction is found to be less than 1.67×10^{-5}. The limits for the polar ellipticity are 1/297.19, 1/297.06, or 1/296.50, 1/296.37 in the second approximation. *W. S. Jardetzky.*

Schüepp, H. Die graphische Lösung des Doppelsternproblems. Elemente der Math. **6**, 33–38 (1951).

Observations of a visual double star system give (1) the normal projection on the celestial sphere of the position of one component relative to the other, (2) the positions of the two components relative to a field of background stars. From five observed positions the true relative ellipse may be constructed, while three positions suffice to determine the center of mass. These two problems are solved by the author with the use of methods of projective geometry.

D. Brouwer (New Haven, Conn.).

Eichhorn, Heinrich. Über Funktionaldeterminante und Ausnahmefälle bei der Bahnbestimmung in der Ellipse. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **158**, 203–225 (1 plate) (1950).

From a mathematical point of view the problem of determining the six elements of the orbit of a planet or comet from three observations is the solution of six equations with six unknowns. Let $y_1, \cdots, y_6$ be the elements to be determined, $x_1, \cdots, x_6$ the observed geocentric spherical coordinates of the object and the three times of the observations, and $x_{10}, \cdots, x_{18}$ the coordinates of the earth relative to the sun. The problem may then be formulated by the six equations: $F_i(y_1, \cdots, y_6, x_1, \cdots, x_{18})=0$, $i=1, \cdots, 6$. If in a point $P_0(y_1, \cdots, y_6, x_{1,0}, \cdots, x_{18,0})$ the Jacobian

$$\frac{\partial(F_1, F_2, F_3, F_4, F_5, F_6)}{\partial(y_1, y_2, y_3, y_4, y_5, y_6)} \neq 0,$$

the solution is determinate. The article presents successively: (1) the derivation of the functions $F_1, \cdots, F_6$; (2) the explicit development of the Jacobian, in which the eccentric anomalies at the observed times are introduced to simplify the expressions; (3) the discovery of exceptional cases for which the Jacobian vanishes. *D. Brouwer.*

Bucerius, H. Bahnbestimmung als Randwertproblem. I. Astr. Nachr. **278**, 193–203 (1950).

Bucerius, H. Bahnbestimmung als Randwertproblem. II. Astr. Nachr. **278**, 204–216 (1950).

The problem of determining an orbit in the two-body problem, when the relative position vector P_1P_2 is given for two distinct values of t, is formulated in terms of integral equations. Various methods of successive approximations for solution of the equations are described and estimates are given for rapidity of convergence. Computational schemes are suggested and particular examples are worked out.

W. Kaplan (Ann Arbor, Mich.).

***Kopal, Zdeněk. The Computation of Elements of Eclipsing Binary Systems.** Harvard Observatory Monograph no. 8. Harvard University Press, Cambridge, Mass., 1950. 181 pp.

The great improvement in accuracy of the observation of the light intensities of stars in recent years has made it desirable to include numerous secondary effects in the interpretation of light curves of eclipsing variables. The author's earlier volume [An Introduction to the Study of Eclipsing Variables, Harvard Observatory Monograph no. 6 (1946); these Rev. **8**, 176] gave an exposition of the theory but the application to the interpretation of light curves was then in an experimental stage. This new volume aims at providing a manual for the analysis of light curves with all the refinement for which theory and observation are adequate.

The procedure followed is in general: (I) Derivation of the preliminary elements. (II) Derivation of intermediary elements, i.e. the derivation of the best possible set of elements, obtained by fitting the light curve to a schematic model. In this discussion the components are still considered either spherical or two similar ellipsoids, with arbitrary limb darkening. (III) Derivation of the final elements. At this stage

of the deviations from the intermediary model the effects of gravity darkening, the dissimilarity of the components in form, as well as the mutual reflections of the two components are introduced. Much attention is given to the discussion of residuals as a means of evaluating the uncertainty of the elements obtained. Useful features of the volume are extensive lists of references at the end of each chapter and an appendix on the method of least squares.

D. Brouwer (New Haven, Conn.).

Wähnl, Maria. Eine theoretische Untersuchung zur Entstehungshypothese der Sternhaufen. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. 158, 227–259 (1950).

The dissolution of a two-body Keplerian orbit in an external field of force is considered. The center of mass of the two bodies is assumed to describe a definite orbit in the external field while the elements of the orbit are considered to undergo secular changes under the influence of the tidal field; these latter changes will be governed by the standard equations of celestial mechanics giving the variation of the orbital elements in the presence of a disturbing function. However, to obtain the secular changes in the orbital elements, the author differentiates the variational equations once more and replaces the periodic terms in the mean anomaly by their respective averages. These changes in the orbital elements are then integrated over the trajectory described by the center of mass of the system. The particular case of the external field considered by the author is that of a "globular cluster" in which the gravitational potential follows a law of the form $(r/R_0)/[3+(r/R_0)^2]^{\frac{1}{2}}$ where (r/R_0) is a measure of the distance from the center in terms of the radius of the cluster, R_0. In the numerical examples worked out, the binary star is assumed to pass centrally through the cluster. The resulting changes in the orbital elements are integrated numerically. When $R_0=15$ parsecs, a double star orbit with a semi-major axis of 100 astronomical units and combined mass of 1 solar mass, suffers a change in eccentricity of amount 6×10^{-17} and a fractional change in semi-major axis of amount 6×10^{-18}; all this takes place in a period of 7×10^{8} years. The implications of this result for the larger problem of the dissolution of star clusters in the general galactic field are briefly discussed.

S. Chandrasekhar (Williams Bay, Wis.).

Camm, G. L. Self-gravitating star systems. Monthly Not. Roy. Astr. Soc. 110, 305–324 (1950).

The equations governing a stellar system stratified in parallel planes and in equilibrium under its own gravitations and in a steady state are:

$$w\partial f/\partial z+(d\Omega/dz)(\partial f/\partial w)=0 \tag{1}$$

and

$$d^2\Omega/dz^2=-4\pi G\rho(z), \tag{2}$$

where $f(z, w)$ is the number of stars per unit interval in velocity w, and per unit height at z, in a column of unit cross section; Ω is the gravitational potential; $\rho(z)=m\int_{-\infty}^{+\infty}f(z, w)dw$ is the density of stars at height z; G is the constant of gravitational attraction and m is the mass of a star. The author shows that equations (1) and (2) admit the following special solutions:

$$\text{(I)}\quad f(z, w)=\frac{\rho_0}{(2\pi)^{\frac{1}{2}}mh}\exp\{-\{\tfrac{1}{2}w^2-\Omega\}/h^2\},$$

$$\text{(II)}\quad f(z, w)=\frac{\rho_0}{m\pi^{\frac{1}{2}}}\frac{(n+\frac{1}{2})!}{n!}(2nh^2)^{n+1}(w^2-2\Omega)^{-n-1}\qquad(n>0)$$

and

$$\text{(III)}\quad f(z, w)=\frac{\rho_0}{m\pi^{\frac{1}{2}}}\frac{(n-1)!}{(n-\frac{1}{2})!}(2nh^2)^{1-n}(2\Omega-w^2)^{n-1}\qquad(n>\tfrac{1}{2}),$$

where ρ_0 is the density of stars at the level $z=0$ and h is a constant. The corresponding expressions for the potential and density are:

$$\Omega=-2h^2\log\cosh\left\{\frac{z}{h}(2\pi G\rho_0)^{\frac{1}{2}}\right\},$$

$$\rho(z)=\rho_0\operatorname{sech}^2\left\{\frac{z}{h}(2\pi G\rho_0)^{\frac{1}{2}}\right\}$$

for case (I) and

$$\Omega=-nh^2\sec^{2/n}\theta;\qquad z=\left(\frac{h^2}{2\pi G\rho_0}\right)^{\frac{1}{2}}\int_0^{\theta}\sec^{1+2/n}\phi\,d\phi,$$

$$\rho(z)=\rho_0\cos^{2(1+1/n)}\theta\qquad(n>-\tfrac{1}{2})$$

for cases (II) and (III); in the latter set of equations n can be positive (case II) or negative (case III). The distribution (I) separates the distributions (II) and (III) and corresponds to the density distribution of an isothermal gas in equilibrium under its own gravitation and is of infinite extent. The distribution (II) is also infinite in extent and corresponds mathematically to the equilibrium configuration for a gas for which the ratio of the specific heats γ is less than unity. And finally the distribution (III) has a finite thickness $[=h(1/n-1)!(-\tfrac{1}{2})!/(1/n-\tfrac{1}{2})!(2\pi G\rho_0)^{\frac{1}{2}}]$ and corresponds to the gas obeying the adiabatic law, $p\propto\rho^{\gamma}$, for $1<\gamma<3$. On comparing the foregoing distributions with the observed distribution of stars in our galaxy perpendicular to the galactic plane, the author finds that the distribution (II) with $n=2/7$ gives the best fit. *S. Chandrasekhar.*

Lindblad, Bertil. On a theorem in the dynamics of stellar systems. Stockholms Observatoriums Annaler 16, 1–34 (1950).

The motion of a star describing a nearly circular orbit in (or near) the "equatorial plane" of a homogeneous oblate spheroid is considered. If the motions are referred to a frame of reference rotating (in the equatorial plane) with the constant angular velocity ω_c of a circular orbit and ξ, η and ζ are the co-ordinates in the radial, the transverse and in the z-directions respectively, then in this rotating frame, the equations of motion are

$$\text{(1)}\quad \ddot{\xi}-2\omega_c\dot{\eta}=0,\quad \ddot{\eta}+2\omega_c\dot{\xi}=0,\quad\text{and}\quad \ddot{\zeta}=\text{constant},$$

where the dots denote differentiation with respect to time. The solutions of these equations can be readily written down. The author considers the perturbation of these equations by a time dependent potential of the form

$$\phi_1=(a_0+a_1\xi+a_2\eta+a_3\zeta+\tfrac{1}{2}a_4\xi^2+\tfrac{1}{2}a_5\eta^2+\tfrac{1}{2}a_6\zeta^2)e^{\nu t},$$

where $a_0, \cdots, a_6$ are constants and ν may be complex. This perturbation introduces terms $\partial\phi_1/\partial\xi$, $\partial\phi_1/\partial\eta$ and $\partial\phi_1/\partial\zeta$, respectively, on the right-hand sides of the equations (1); the corresponding first order changes in the solutions can be written down. For $|\nu|\to0$, it is found that the fractional increments in the velocities $\dot{\xi}$ and $\dot{\eta}$ are the same. Consequently, if the initial distribution of $\dot{\xi}$ and $\dot{\eta}$ is Gaussian with a dispersion α the new distribution will also be Gaussian with a dispersion $\alpha_1=\alpha(1+\delta)$ where δ may be a function of time. On the assumption that the density in the phase space is unaltered by the perturbation, it is concluded that the density per unit area in the (ξ, η)-plane is altered in the

ratio α_1^2/α^2. Similarly if the distribution of velocities in the ζ-direction is initially Gaussian with a dispersion γ, it remains Gaussian with a dispersion γ_1 (say) after the perturbation and the density is altered in the inverse ratio of the γ's. Combining these results we have: if ρ_0 and ρ denote, respectively, the densities before and after the perturbation, then $\rho/\rho_0=(\gamma_1/\gamma)(\alpha_1^2/\alpha^2)$. This is the theorem referred to in the title of the paper. The author states: "The validity under fairly general conditions of a generalized adiabatic law, governing the relation between internal velocity dispersion and density, appears to be of fundamental importance for the dynamics of stellar systems. It should be emphasized that the adiabatic changes of velocity dispersion are here in no way consequences of encounters between individual stars, but follow from the general dynamics of the condensation itself. . . . This theorem will serve as basis for a renewed investigation of the mass motions in stellar systems according to principles set forth in previous papers. It turns out that the changes involved by assuming 'adiabatic' instead of 'isothermal' variations do not affect the essential results of the previous analysis." [In view of the fact that the theorem has been established only on the explicit assumption of a Gaussian distribution of the velocities, it is doubtful whether the theorem is as general as the author claims.] *S. Chandrasekhar.*

Schatzman, Evry. Le spectre des naines blanches et leur débit d'énergie. Danske Vid. Selsk. Mat.-Fys. Medd. **25**, no. 7, 100 pp. (1950).

In this paper the following problems relating to the structure and constitution of the white dwarf stars are considered: (i) the statistical mechanics of the separation of the elements according to their atomic weights (particularly hydrogen and helium) in a strong gravitational field, (ii) the effect of convective instability in the outer layers in restoring the uniformity of the composition in these layers, (iii) the structure of the atmospheres of these stars, and (iv) the production of energy by the proton-proton reaction in the outer envelope as the source of their luminosity.

S. Chandrasekhar (Williams Bay, Wis.).

Bondi, C. M. Models for red giant stars. I. General discussion and application to homogeneous models. Monthly Not. Roy. Astr. Soc. **110**, 275–286 (1950).

In this paper stellar models of uniform chemical composition consisting of radiative envelopes and convective cores are discussed in terms of the homology invariant variables $S=4\pi Pr^4/GM^2(r)$, $Q=Pr/GM(r)\rho$ and $N=1-d\log\rho/d\log P$ (where the various symbols have their usual meanings) which were introduced by the author in an earlier paper [same Not. **109**, 62–85 (1949); these Rev. 11, 695]. It is shown that for these models, the allowed range in M/R (mass/radius) for the configurations is not large; from this it is concluded that these models cannot provide a basis for accounting for "giant stars" which are believed to be characterized by large values M/R. *S. Chandrasekhar.*

Bondi, C. M., and Bondi, H. Models for red giant stars. II. Models with a chemical inhomogeneity and opacity due to photoelectric effect. Monthly Not. Roy. Astr. Soc. **110**, 287–304 (1950).

In this paper the authors consider stellar models consisting of (i) a convective core (having polytropic distribution with $n=1.5$), (ii) an intermediate radiative zone with a composition equal to that of the core, and (iii) a radiative envelope differing in composition from the core. In addition, the following assumptions are made: (a) The transition in chemical composition occurs discontinuously at a spherical interface; (b) from the center to the interface the composition is uniform; similarly, from the interface to the surface of the star the composition is also uniform; (c) the law of opacity is $\kappa=\kappa_0\rho^{0.75}T^{-3.5}$ (κ_0 a constant); and (d) the entire generation of energy from nuclear sources occurs within the convective core. The authors discuss the problem in terms of their variables S, Q, and N [see the preceding review] which satisfy the pair of equations

$$\frac{dQ}{dS}=\frac{Q}{S}\frac{S+N-Q}{1+2S-4Q} \quad \text{and} \quad \frac{dN}{dS}=\frac{N}{S}\frac{S+1.75-8.25N}{1+2S-4Q}.$$

They show that under certain conditions models having a relatively high value for M/R (mass/radius) can be constructed: This depends on whether in the (S, Q)-plane the solutions spiralling round the singular point $S=0$, $Q=N=7/33$ (where both dQ/dS and dN/dS are zero) are used to describe the outer radial envelope of the model.

S. Chandrasekhar (Williams Bay, Wis.).

ten Bruggencate, P. Bemerkungen über den Zusammenhang zwischen dem Rosseland'schen und dem Chandrasekhar'schen Wert für die Opazität in Sternatmosphären. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1950**, 7–13 (1950).

For purposes in the theory of stellar atmosphere the continuous absorption coefficient κ_ν has been averaged in one of two ways: The "Rosseland mean" given by

$$\frac{1}{\bar{\kappa}}=\int_0^\infty \frac{1}{\kappa_\nu}\frac{dK_\nu}{dT}d\nu \Big/ \int_0^\infty \frac{dK_\nu}{dT}d\nu,$$

where T denotes the temperature and $K_\nu=\frac{1}{2}\int_{-1}^{+1}I_\nu\mu^2 d\mu$ (I_ν is the specific intensity of the radiation in frequency ν and in a direction making an angle $\cos^{-1}\mu$ with the outward normal) and the "Chandrasekhar mean" given by

$$\bar{\kappa}=\frac{1}{F}\int_0^\infty \kappa_\nu F_\nu d\nu,$$

where F_ν is the net monochromatic flux in ν. In the framework of an exact theory the two means are identical. In practise, however, in using the Rosseland mean, K_ν is replaced by the Planck intensity B_ν (the "reasoning" being $K_\nu\sim\frac{1}{3}J_\nu\sim\frac{1}{3}B_\nu$), and in using the Chandrasekhar mean the monochromatic flux F_ν [1] in a grey atmosphere is used. It is well known (for example, in the theory of stellar interiors) that the Rosseland mean is the proper one to use whenever the net flux F is very small compared to the mean intensity J. The author restates this argument in a different form and presents it as a case for the Rosseland mean as against the Chandrasekhar mean. [The reviewer is unable to agree with this assessment of the situation: In the outer parts of a stellar atmosphere $F\sim J$ and the assumption that K_ν can be replaced by B_ν is essentially equivalent to the assumption that we have monochromatic radiative equilibrium and ignores the essential process which occurs in local thermodynamic equilibrium, namely, the transformation of the radiation into high frequencies as we descend into the atmosphere. Moreover, even in a grey atmosphere K_ν differs from $\frac{1}{3}J_\nu$ and this again differs from $\frac{1}{3}B_\nu$; these differences are by no means negligible.] *S. Chandrasekhar.*

Hitotuyanagi, Zyuiti. Über die Randverdunkelung der Sonne. I. Zweite Näherungslösung in der Milne-Lindbladschen Theorie von Blanketing Effekt der umkehrenden Schicht. Jap. J. Astr. Geophysics 19, 97–111 (1941).

In considering the effect of the crowding of the absorption lines on the continuous spectrum and the temperature distribution in a stellar atmosphere (the so-called blanketing effect) one sometimes adopts a suggestion of Milne that at $\tau=0$ one uses the boundary condition; the inward flux of radiation is a specified fraction η of the outward flux of radiation instead of the usual one of the absence of incident radiation. The solution to the corresponding transfer problem has been known on the Milne-Eddington approximation in which one puts $J=\int_{-1}^{+1}Id\mu=3\int_{-1}^{+1}I\mu^2d\mu=3K$. In a higher approximation one may use the source function given by this approximation to compute the radiation field, determine the ratio $J/3K$ as a function of τ, and use this as a basis of a higher approximation. This method is adopted in this paper to obtain a higher approximation to the solution of the blanketing problem. *S. Chandrasekhar.*

Hitotuyanagi, Zyuiti. Über die Randverdunkelung der Sonne. II. Der Einfluss der Fraunhoferschen Linien auf das kontinuierliche Spektrum. Jap. J. Astr. Geophysics 19, 113–134 (1941).

As an alternative to Milne's suggestion of treating the blanketing effect [see preceding review] Chandrasekhar [Monthly Not. Roy. Astr. Soc. 96, 21–42 (1935); see also G. Münch, Astrophys. J. 107, 87–109 (1946)] introduced the assumption that the absorption lines occur uniformly over the whole spectral range with a probability $a_2<1$. In the spectral lines there is the effect of both line and continuous absorption coefficients; outside of the spectral lines we have the effect only of the continuous absorption coefficient. The solution for the problem when the ratio of the total absorption coefficients inside and outside of the spectral lines is a constant, is known. In this paper the author solves the same problem when that ratio varies with depth in the manner $\alpha+\beta e^{-\lambda\tau}$ where α, β and λ are constants and τ is the optical thickness in the continuum. The theory is illustrated by a number of applications to practical solar problems. *S. Chandrasekhar* (Williams Bay, Wis.).

Hagihara, Yusuke, and Hatanaka, Takeo. Radiative transfer in a planetary nebula. Jap. J. Astr. Geophysics 19, 135–216 (1942).

In treating the transformation of the incident ultraviolet radiation in a planetary nebula beyond the head of the Lyman series (frequency ν_0 say) into radiation in the Lyman and the Balmer series, one generally ignores the dependence on wave-length of the absorption coefficient beyond ν_0 and assigns to the entire spectral range a mean absorption coefficient. However, according to the physical theory the absorption coefficient beyond ν_0 must fall off $(\nu_0/\nu)^3$ ("Kramers' Law") and in this paper an attempt is made to allow for this variation. First the total density in the ultraviolet radiation is determined by assigning to it a mean frequency and a mean absorption coefficient and solving the equation of transfer in the Milne-Eddington approximation; this is then used in the equation of transfer for the separate frequencies. In this way an iteration scheme is worked out. The paper is full of many detailed calculations and appropriate tables for the astrophysical applications contemplated.

S. Chandrasekhar (Williams Bay, Wis.).

***Alfvén, H. Cosmical Electrodynamics.** Oxford University Press, New York, N. Y., 1950. viii+237 pp. $5.00.

This book is a manifestation of the growing interest in electromagnetic phenomena in astrophysics, an interest which was largely stimulated by the author himself in the early 1940's. An introductory chapter in which the principal problems are surveyed, is followed by three chapters dealing with the motion of charged particles in a magnetic field, electric discharges and magneto-hydrodynamic waves. The ideas developed in these chapters are applied in the following three chapters to problems in solar physics (sunspots, the chromosphere, the prominences and the corona), magnetic storms, aurorae and cosmic radiation.

It is safe to say that the permanent parts of the book are included in the first three chapters: And here attention may be called to the methods developed by the author for dealing with the motion of charged particles of moderate energy in a magnetic field. By ignoring the details of the spiralling about the magnetic lines of force, and concentrating on the motion of the "guiding center", the author is able to obtain a broad picture of the phenomena with very few calculations. And in the chapter on electric discharges, the illustrations in terms of equivalent circuits are particularly apt though one wonders whether a treatment in terms of a mixture of Maxwell's equation and equivalent circuits of complicated astrophysical phenomena is not beset with its own dangers. And in his treatment of the lines of force, the magnetic lines of force "come to life, writhing and dragging a sticky material about with them". The astrophysical chapters are speculative and the author himself would not claim that any of the large problems he considers have received their final and definite solutions. *S. Chandrasekhar.*

RELATIVITY

***Dive, Pierre. Ondes ellipsoïdales et relativité.** Gauthier-Villars, Paris, 1950. x+140 pp.

In the first part of this book the author proposes to avoid the "paradoxes and contradictions" of the special theory of relativity by introducing a privileged Galilean reference frame S with coordinates x, y, z, t and relating Galilean frames S' with coordinates x', y', z', t' which move relative to these by the coordinate transformations $x=ut+x'$, $y=y'$, $z=z'$, $t'=at+bx$, where u is the velocity of S' relative to S, taken to be along the x axis, a and b are functions of u, and the x, y, z axes of S are coincident with the x', y', z' axes of S' at time $t=0$. It follows that a spherically symmetric light signal emitted in the system S' at time $t'=0$ when referred to the time t has a wave front which is an ellipsoid of revolution whose eccentricity depends on the functions $a(u)$ and $b(u)$. These functions are partially determined by the author by appealing to the results of various optical experiments. In the second and last part of the book the author shows that the general theory of relativity admits solutions for the line element which are of the type postulated for the privileged Galilean reference frame when the stress energy tensor is restricted to be of a prescribed form.

H. P. Robertson has shown [Rev. Modern Physics 21, 378–382 (1949); these Rev. 11, 215] that the transformation relating two moving observers must be the Lorentz transformation if one synchronizes clocks at different places by

means of light signals and if one accepts the results of the Michelson-Morley, Kennedy-Thorndike, and Ives-Stillwell experiments. In view of Robertson's results the transformation proposed by the author cannot be taken seriously, for it either disagrees with these experiments or depends on a synchronization scheme which cannot be carried out in practice as it involves a nonoperational appeal to the priveleged Galilean reference frame. *A. H. Taub.*

Janković, Zlatko. Le principe variationnel de la théorie de la relativité restreinte. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **5**, 12–20 (1950). (Croatian. French summary)

The paper discusses the well-known principles of classical dynamics, and its author shows that it is possible to deduce some of the results of the special theory of relativity on a different axiomatic basis from that conventionally adopted. He points out, in particular, that the principle of d'Alembert, which is Lorentz-invariant in a four-dimensional space, can be used as a basis for deducing different systems of mechanics which have some features in common with the special relativity, and become identical with the classical mechanics in a limit. The author makes no comment on the relation between his postulates and Milne's theory of kinematic relativity. *Z. Kopal* (Cambridge, Mass.).

Ives, Herbert E. Revisions of the Lorentz transformations. Proc. Amer. Philos. Soc. **95**, 125–131 (1951).

The author claims that the special theory of relativity and, in particular, Einstein's principle of the constancy of the velocity of light, are logically inconsistent. He derives a complicated set of "revised Lorentz-Poincaré transformations" in which all variables are defined operationally. However, by a change of variables, the author's transformation equations reduce to the ordinary Lorentz transformation.
A. Schild (Pittsburgh, Pa.).

Fink, K. Metrisches Feld und skalares Materiefeld. Comment. Math. Helv. **25**, 26–42 (1951).

Using the Mie-Hilbert formulation of the relation of the metrical and matter fields in terms of a variation problem, the field equations of general relativity are discussed for the spherically symmetric case under the assumption that the energy-momentum tensor can be derived from a single scalar function S, as follows

$$T_{\mu\nu}=k[2\delta_\mu{}^\rho\delta_\nu{}^\sigma-g_{\mu\nu}g^{\rho\sigma}]S_{,\rho}S_{,\sigma},$$

where k is a constant, $g_{\mu\nu}$ is the metric tensor and the comma means partial differentiation. Explicit expressions for S and $g_{\mu\nu}$, satisfying the field equations, are given. Three cases are distinguished two of which give infinite total energy for the universe, the third gives finite negative total energy. A concluding section studies the cosmological problem when spatial isotropy is imposed on the above solutions.
A. J. Coleman (Toronto, Ont.).

Ingraham, Richard. Relativité conforme. C. R. Acad. Sci. Paris **232**, 1072–1074 (1951).

Ingraham, Richard. Sur une théorie de la "relativité conforme." C. R. Acad. Sci. Paris **232**, 938–940 (1951).

In the first of these papers the author gives a variational method for determining a pair of conformally invariant linear connections which he proposes to use for a unified field theory. In the second paper the author specializes the transformation laws of the geometric objects in the two spaces he had previously considered and derives a set of unified field equations which presumably describe the gravitational field, the electromagnetic field, and another (meson) field of zero mass. *A. H. Taub* (Urbana, Ill.).

García, Godofredo. On the unified field. Actas Acad. Ci. Lima **13**, 17–27 (1950). (Spanish)

This paper presents certain elaborations of a work of Levi-Civita [A Simplified Presentation of Einstein's Unified Field Equations, Blackie, London-Glasgow, 1929].
W. Kaplan (Ann Arbor, Mich.).

Kurşunoğlu, Behram. Space-time on the rotating disk. Proc. Cambridge Philos. Soc. **47**, 177–189 (1951).

Starting with a line-element

$$ds^2=-dr^2-Fd\theta^2-dz^2+Gdt^2-2Hd\theta dt,$$

with F, G, H, functions of r only, the author transforms it into

$$ds^2=-dr^2-e^\nu d\theta^2-dz^2+e^\mu dt'^2.$$

He introduces an energy-momentum tensor with components

$$T_1{}^1=T_2{}^2=-T_3{}^3=p,\quad T^{44}=(\rho c^2-p)(dt/ds)^2+\varepsilon^{-\mu}p$$

[presumably dt should be read dt'] and solves the field equations to obtain μ and ν, and hence the line-element for the rotating disk

$$ds^2=(c^2-\omega^2r^2)dt'^2-dr^2-r^2(1-\omega^2r^2/c^2)^{-1}d\theta^2-dz^2,$$

where ω is a constant, the angular velocity of the disk. He discusses the intrinsic geometry of the disk and time measurement on it, and in the second part of the paper uses the general method of W. O. Kermack, W. H. McCrea, and E. T. Whittaker [Proc. Roy. Soc. Edinburgh **53**, 31–47 (1933)] to discuss the measurement of distance on the disk and the Doppler effect. *J. L. Synge* (Dublin).

Gilloch, Josephine M., and McCrea, W. H. The relativistic mass of a rotating cylinder. Proc. Cambridge Philos. Soc. **47**, 190–195 (1951).

This paper is based on a paper by G. L. Clark [same Proc. **45**, 405–410 (1949); these Rev. **11**, 281] which the authors interpret as applying to a cylinder whose length is finite but large compared with its radius. They examine the gravitational potentials $\gamma_{\mu\nu}$ of Clark and note certain corrections to his values. They evaluate the Newtonian potential, gravitational mass, gravitational density, energy, density of rest-mass, relative density, and invariant density, and tabulate the various expressions for density arising from the various definitions, for both plane stress and isotropic stress.
J. L. Synge (Dublin).

Schrödinger, E. Studies in the non-symmetric generalization of the theory of gravitation. I. Communications Dublin Inst. Advanced Studies. Ser. A. no. **6**, i+28 pp. (1951).

The Hermitian field equations proposed by Einstein for a nonsymmetric tensor g_{ij} augmented by the requirement that $\Gamma^i_{kl}=0$ are solved by an approximation method. It is assumed that $g_{ij}=\eta_{ij}+\varphi_{ij}+\gamma_{ij}$ where the φ_{ij} represent the Maxwell field and are of the first order and the γ_{ij} represent the gravitational field and are of the second order. The "matter tensor" which determines the gravitational field γ_{ij} is entirely different from that entering in older theories. The author states "the approximation reached here is insufficient and will have to be extended in order to reveal the reaction of the fields on their sources." *A. H. Taub.*

Heller, Jack. Covariant transformation law for the field equations. Physical Rev. (2) 81, 946–948 (1951).

It is assumed that field equations derived from a variational principle have a general transformation law under infinitesimal coordinate transformations. It is then shown that conditions necessary for the formulation of the algebraic relationships between the canonical momenta are satisfied. If the transformation laws of the field equations are restricted, conservation laws which are satisfied even when the field equations do not hold may be determined.

A. H. Taub (Urbana, Ill.).

Zatzkis, Henry. Conservation laws in the general theory of relativity with electromagnetic field. Physical Rev. (2) 81, 1023–1026 (1951).

The author reformulates the conservation laws of the general theory of relativity with an added electromagnetic field, which hold even when the field equations are not satisfied, in terms of relationships between integrals taken over two-dimensional surfaces. The formulation is not invariant and thus different values of the integrals involved are obtained in different coordinate systems.

A. H. Taub (Urbana, Ill.).

Moshinsky, Marcos. On the interactions of Birkhoff's gravitational field with the electromagnetic and pair fields. Physical Rev. (2) 80, 514–519 (1950).

Birkhoff's gravitational equations in flat space-time are extended to include fields other than gravitational by the process of adding terms to the Lagrangian function from which Birkhoff's equations are obtained. One choice of additional terms leads to a generalization of Maxwell's equations. If these are accepted, the choice of additional terms is shown to be consistent with observation in that they lead to the same result for the bending of light-rays in the Sun's gravitational field as does general relativity. A second, and different, choice of additional terms in the Lagrangian leads to a modification of Dirac's equations for the electron and consistency with observation is again secured by obtaining the general relativity formula for the gravitational reddening of light.

G. C. McVittie (London).

Keberle, Edouard, et Mercier, André. Comportement relativiste des systèmes avec ou sans self-contrainte. Arch. Sci. Soc. Phys. Hist. Nat. Genève 3, 235–241 (1950).

Keberle, Edouard. Approximation galiléenne de l'attribution relativiste onde-corpuscule. Arch. Sci. Soc. Phys. Hist. Nat. Genève 3, 242–246 (1950).

Majorana, Quirino. Critica della relatività di Einstein. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 5 (1947/48), 91–114 (1949).

Majorana, Quirino. L'errore della relatività di Einstein. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 6, 81–96 (1950).

MECHANICS

Tzénoff, Iv. Quelques formes nouvelles des équations générales du mouvement des systèmes matériels. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. 45, 239–261 (1949). (Bulgarian. French summary)

Consider a nonholonomic system, and let $k+p$ be the number of independent coordinates

$$q_\alpha,\ q_{\alpha+i} \qquad (\alpha=1, 2, \cdots, k;\ i=1, 2, \cdots, p)$$

required to specify the configuration of the system and k be the number of its degrees of freedom. Suppose that $(*)$ $q'_{k+i}=\sum_\alpha a_{i\alpha}q_\alpha'+a_i$ $(i=1, 2, \cdots, p)$ are the nonintegrable equations connecting the variations of the coordinates; in holonomic systems these equations are of course nonexistent. The kinetic energy T_0 of the system, not taking into account $(*)$, i.e., the kinetic energy of the associated holonomic system, is a function of t, q_α, $q_{\alpha+i}$, q_α', $q'_{\alpha+i}$; its first and second order derivatives T_0' and T_0'' with respect to t involve the derivatives of q_α and $q_{\alpha+i}$ up to the second and third orders respectively. The kinetic energy T of the nonholonomic system and the derivatives T' and T'' are obtained from T_0, T_0' and T_0'' respectively by taking into account $(*)$. Denote by T_1' and T_2', respectively, the function T_0' considered as a function of the q_α', q'_{k+i} and of the q_α'', q''_{k+i} respectively, the derivatives q'_{k+i} and q''_{k+i} being given by $(*)$ and the relation obtained from it by differentiation with respect to t. Similarly, denote by T_2'' and T_3'' respectively the function T_0'', considered as a function of the q_α'', q''_{k+i} and of the q_α''', q'''_{k+i} respectively. The author obtains the following three novel forms for the equations of motion of a nonholonomic system:

$$2\frac{d}{dt}\frac{\partial T}{\partial q_\alpha'}-\frac{\partial T'}{\partial q_\alpha'}+3\frac{\partial T_2'}{\partial q_\alpha'}-2\frac{\partial T_3''}{\partial q_\alpha''}=Q_\alpha,$$

$$2\frac{\partial T''}{\partial q_\alpha''}-3\frac{\partial T'}{\partial q_\alpha'}+3\frac{\partial T_2'}{\partial q_\alpha'}-2\frac{\partial T_3''}{\partial q_\alpha''}=Q_\alpha,$$

$$2\frac{\partial T_2''}{\partial q_\alpha''}-3\frac{\partial T_1'}{\partial q_\alpha'}=Q_\alpha \qquad (\alpha=1, 2, \cdots, k),$$

which become considerably simpler in the holonomic case. The first set of equations may also be put in Lagrange's form if one introduces on the left-hand side of these equations a supplementary term whose expression can be easily deduced. Further, assuming that the derivatives q_α' are linear functions of new variables ω_β, some transformed forms for the equations of motion of both nonholonomic and holonomic systems are obtained. These new equations of motion are applied to the motion of a hoop on a fixed horizontal plane. Two cases are considered: (1) the hoop slides without friction, and (2) it rolls without sliding.

E. Leimanis (Vancouver, B. C.).

Mineur, Henri. Quelques propriétés générales des équations de la mécanique. Bull. Astr. (2) 13, 309–328 (1948).

In connection with a Hamiltonian system with n degrees of freedom and k uniform first integrals in involution, the author considers the so-called generalized equations of mechanics,

$$\frac{\partial q_i}{\partial w_h}=\frac{\partial f_h}{\partial p_i},\quad \frac{\partial p_i}{\partial w_h}=-\frac{\partial f_h}{\partial q_i}, \qquad i=1, \cdots, n;\ h=1, \cdots, k.$$

Here $f_1, \cdots, f_k$ are the first integrals and $w_1, \cdots, w_k$ are a new set of independent variables. These generalized (partial differential) equations are completely integrable in virtue of the fact that the f's are in involution, and it is then shown how the general solution of the original dynamical problem can be readily deduced from the general solution of the generalized equations. The author then extends the Jacobi theory and the theorem of Liouville to the generalized equations. A parametric theory of the same type is developed in case the Hamiltonian and the first integrals are given implicitly by means of k equations. Finally the case $k=n$, when of course it is well known that the original system can be integrated by quadratures, is considered in special detail. *D. C. Lewis* (Baltimore, Md.).

Mineur, Henri. Réduction d'une forme quadratique dans le groupe linéaire canonique. Bull. Astr. (2) **15**, 107–141 (1950).

A proof is given of the theorem that any quadratic Hamiltonian H in $p_1, \cdots, p_n, q_1, \cdots, q_n$ can be reduced by means of a canonical transformation to a sum of Hamiltonians of the form $\lambda \sum_{i=1}^{n} p_i q_i + \sum_{i=1}^{n-1} q_i p_{i+1}$ where the λ's are latent roots (not necessarily distinct) of the matrix of coefficients of the original Hamiltonian. The case of distinct λ's is classical, but the general case apparently has never before been adequately treated. The author uses this reduction to establish the existence of n first integrals in involution for linear Hamiltonian systems. *D. C. Lewis.*

Mendes, Marcel. Sur les transformations canoniques. Bull. Astr. (2) **15**, 307–316 (1950).

An elaboration of the author's previous note [C. R. Acad. Sci. Paris **226**, 1240–1242 (1948); these Rev. **9**, 540]. *D. C. Lewis* (Baltimore, Md.).

Nadile, Antonio. Influenza di vincoli anolonomi sullo spostamento di equilibrio di un sistema. Rivista Mat. Univ. Parma **1**, 393–399 (1950).

The displacement of an equilibrium configuration due to the introduction of holonomic constraints has been discussed by Levi-Civita [Lezioni di meccanica razionale, vol. 2, part 1, Zanichelli, Bologna, 1926, p. 447]. An analogous discussion involving nonholonomic constraints is given here, including an investigation of stability. *D. C. Lewis.*

Saltykow, N. Application de la théorie des groupes fonctionnels au problème des trois corps. Bull. Astr. (2) **15**, 293–305 (1950).

Resuming a study undertaken by Lie [Math. Ann. **8**, 215–303 (1875)] and Engel [Die Liesche Theorie der partiellen Differentialgleichungen erster Ordnung, Teubner, Leipzig-Berlin, 1932, especially pp. 361–362], the author determines two sets of integrals in involution of the three-body problem. The first set contains five functions, the second six functions. These sets of functions are used to reduce the problem to the solution of systems of partial differential equations of first order; the two systems are formed of 6 and 7 equations respectively. *W. Kaplan.*

Sémirot, Pierre. Stabilité des solutions périodiques dans le problème des deux centres fixes. Bull. Astr. (2) **14**, 37–60 (1949).

The equations of variation for the problem of two fixed centers (of Newtonian gravitational attraction) are written down and the characteristic exponents are systematically studied for the various possible periodic motions. Since the fundamental differential equations admit three first integrals, the characteristic exponents are in general all zero, but there are certain cases where this is not the case because of the vanishing of certain functional determinants. Even in these exceptional cases, the solutions turn out to be unstable in all cases considered by the author. The generality of his conclusions, however, is somewhat limited by the fact that certain power series in a parameter ϵ which he was obliged to introduce were known to converge only for $|\epsilon|$ sufficiently small. *D. C. Lewis* (Baltimore, Md.).

Sémirot, Pierre. Solutions périodiques et stabilité dans le mouvement pendulaire. Bull. Astr. (2) **13**, 253–256 (1948).

The equations of variation for the simple pendulum problem are written down, solved, and discussed in trivial illustration of Poincaré's theory of characteristic exponents. *D. C. Lewis* (Baltimore, Md.).

Kasner, Edward, and De Cicco, John. Physical families in the gravitational field of force. Amer. Math. Monthly **58**, 226–232 (1951).

In previous papers [Proc. Nat. Acad. Sci. U. S. A. **34**, 68–72 (1948); **35**, 106–108, 201–204, 419–422 (1949); these Rev. **9**, 375; **10**, 398, 631, 747], the authors have defined the concept of physical families of curves in positional fields of force, and have discussed general properties of such families. In the present paper they consider the case in which the force is constant in magnitude and direction. It is shown that any physical family of curves in such a field of force can be found by means of a quadrature. The particular families for which the quadrature can be effected in terms of elementary functions are determined. *L. A. MacColl.*

Bundgaard, Svend. On the motion of a heavy particle on a smooth surface of revolution. Mat. Tidsskr. B. **1950**, 63–65 (1950). (Danish)

Horizontal and vertical velocity components are computed, and some inequalities are derived. An investigation is made of the highest and lowest levels the particle reaches after a finite or infinite lapse of time. Stationary states are discussed. *J. A. Schouten* (Epe).

Bottema, O., and Beth, H. J. E. Euler's equations for the motion of a rigid body in n-dimensional space. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 106–108 (1951).

A certain system of $\frac{1}{2}n(n-1)$ equations is given as the appropriate generalization of Euler's well-known equations for a rigid body in 3-dimensional space. *D. C. Lewis.*

Bottema, O. On the stabilization of equilibrium by rotation. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 61–65 (1951).

A generalization to four dimensions of some theorems of Brouwer, one of which may be formulated as follows: A particle P moves under the influence of gravity on a smooth surface Ω which is rotating with uniform velocity ω about a vertical axis l, which intersects Ω in a point O of horizontal tangency. If the principal curvatures k_1 and k_2 of Ω at O are both positive then O is a stable equilibrium point for small values of ω^2 and also for large values of ω^2, but there is (for $k_1 \neq k_2$) a region for ω^2 where the equilibrium is unstable. *D. C. Lewis* (Baltimore, Md.).

Stoppelli, Francesco. Sul principio dell'effetto giroscopico. Giorn. Mat. Battaglini (4) 4(80), 14–38 (1951).

Signorini has established the validity of certain simple approximations in the theory of the motion of a gyroscope [Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (8) 1, 1–41 (1946); these Rev. 9, 109]. In the present paper the author investigates the extent to which similar approximations can be used in more general problems concerning a rigid body moving, with one point fixed, under the influence of external forces. Appropriate generalizations of Signorini's results are obtained, but they cannot be summarized easily in a way which is both concise and explicit.

L. A. MacColl (New York, N. Y.).

Zeuli, Tino. Sul moto di un corpo rigido pesante asimmetrico col baricentro appartenente all'asse di una delle sezioni circolari dell'ellissoide d'inerzia. Univ. e Politecnico Torino. Rend. Sem. Mat. 9, 263–295 (1950).

The author treats the problem described in the title by expanding the unknown functions in powers of a parameter, the coefficients being functions of the time. The parameter is such that its vanishing reduces the problem to the classical case of the symmetrical gyroscope. *D. C. Lewis.*

Griseri, Bruna. Soluzioni meromorfe delle equazioni del moto di un solido pesante intorno a un punto fisso col baricentro situato sull'asse di una delle sezioni circolari dell'ellissoide d'inerzia relativo al punto fisso. Univ. e Politecnico Torino. Rend. Sem. Mat. 9, 225–243 (1950).

Solutions of the problem described in the title which are meromorphic in the plane of the complex variable t with a pole at the origin are studied by the author. Two such solutions always exist and a third is also possible provided that the constants in the original equations satisfy a certain condition. *D. C. Lewis* (Baltimore, Md.).

Manacorda, T. Sul moto di un solido planare intorno ad un punto fisso. Rivista Mat. Univ. Parma 1, 383–391 (1950).

The equations of motion of a heavy rigid body with a fixed point are written down on the assumption that the center of gravity lies on one of the principal planes of inertia relative to the fixed point. The author then finds two invariant relations in the form of expressions for the components of angular velocity as power series in a certain linear combination of these components. He thus arrives at a representation for the general solution and exhibits how a special case of a partial solution by quadratures (effected by Hess) fits into the theory. *D. C. Lewis* (Baltimore, Md.).

Nadile, Antonio. Sul moto intorno a un punto fisso di un corpo rigido pesante il cui baricentro appartiene a uno dei piani principali di inerzia. Matematiche, Catania 5, 68–82 (1950).

The author finds two types of uniform rotation in the problem of the motion of a rigid heavy body about a fixed point with center of gravity located on one of the principal planes of inertia relative to the fixed point. Certain motions involving only quadratures and the integration of a Riccati equation are also found. These latter depend on five constants of integration. *D. C. Lewis* (Baltimore, Md.).

Aronovič, G. V. On the theory of shimmy of the automobile and the airplane. Akad. Nauk SSSR. Prikl. Mat. Meh. 13, 477–488 (1949). (Russian)

By the shimmy of an automobile or an airplane with tricycle landing gear is meant an oscillation of the front wheels around the vertical. This oscillation is actually a self-excited one and nonlinear. What matters primarily is to determine the conditions under which shimmy may arise. Most authors, Rocard excepted, have discussed the problem disregarding the influence of the motion of the vehicle as a whole. In the present paper the complicated system of differential equations for the full motion is set up both for an automobile and a plane. The small oscillations for shimmying are discussed through the equations of the first approximation (a linear system).

For an automobile the following two special cases are examined at length: (a) the vehicle remains strictly horizontal and under rectilinear motion; (b) the wheels are laterally rigid. In the treatment of the plane similar simplifying assumptions are introduced. [Relevant references: Becker, Fromm, and Maruhn, Schwingungen in Automobillenkungen, Berlin, 1931; Rocard, Revue Sci. 84, 15–28 (1946); these Rev. 8, 69; D. Sensaud de Lavaud, C. R. Acad. Sci. Paris 185, 254–257 (1927); F. Weick, J. Soc. Automotive Engrs. (Trans.) 68, 176–187 (1936); Kantrowitz, Tech. Rep. Nat. Adv. Comm. Aeronaut., no. 686 (1940); Wylie, J. Aeronaut. Sci. 7, 56–67 (1939); Andronow and Chaikin, Theory of Oscillation, translation published by Princeton University Press, Princeton, N. J., 1949; these Rev. 10, 535; Julien and Rocard, La stabilité de route des locomotives, Actual. Sci. Ind., no. 279, Hermann, Paris, 1935; Yu. I. Neĭmark, Avtomatika i Telemehanika 9, 190–203 (1948); these Rev. 12, 498; Doklady Akad. Nauk SSSR (N.S.) 59, 853–856 (1948); these Rev. 9, 428; the paper listed below.]

S. Lefschetz (Princeton, N. J.).

Metelicyn, I. I. Concerning the oscillation of a wheel with an elastic tire. Doklady Akad. Nauk SSSR (N.S.) 61, 449–452 (1948). (Russian)

de Castro Brzezicki, A. On the movement of a point of variable mass. Revista Mat. Hisp.-Amer. (4) 10, 233–237 (1950). (Spanish)

The equations of motion of a mass particle whose mass decreases exponentially with time, the decrease being expelled with constant absolute velocity, are solved by an iterative procedure.

P. Franklin (Cambridge, Mass.).

Paleček, E. M. On the computation of the trajectories of the centers of gravity of artillery shells. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 83–96 (1951). (Russian)

The essence of Kazakov's method [Akad. Nauk SSSR. Prikl. Mat. Meh. 9, 129–138 (1945); these Rev. 7, 139] for numerical integration of the system of differential equations of the particle problem in exterior ballistics lies in the fact that the extrapolations are not directly performed upon the one or the other variables, but rather upon certain auxiliary quantities which have little influence on the results of extrapolation. Kazakov himself used as the independent variables the abscissa of the center of gravity and the time of flight of the particle. The author extends Kazakov's method to the case where either the tangent of the angle of inclination of the tangent to the trajectory or the horizontal component of the velocity is taken as the independent variable. Estimates of the relative error of the results for each choice of the independent variable are given, and the appropriateness of the choice to the purpose is discussed.

E. Leimanis (Vancouver, B. C.).

Schmidt, W. Untersuchungen des für den schiefen Stoss elastischer Kugeln gültigen Reflektionsaxioms und einige Folgerungen daraus. Z. Angew. Math. Mech. **30**, 182–184 (1950).

The author studies the oblique incidence of a sphere on a rigid barrier for both elastic and inelastic impact. He claims that when rotation is considered the reflection axiom (the angle of incidence equals the angle of reflection) is no longer valid. It is not clear, at least in the elastic case, how a second solution to the quadratic equations involved is eliminated. This additional solution would yield the reflection law and thereby contradict the author's claim.

G. H. Handelman and A. Schild (Pittsburgh, Pa.).

Hamel, G., und Schmidt, Wolfgang. Zu W. Schmidt, Untersuchungen des für den schiefen Stoss elastischer Kugeln gültigen Reflektionsaxioms und einige Folgerungen daraus. Z. angew. Math. Mech. 30 (1950), S. 182 bis 184. Z. Angew. Math. Mech. **31**, 191–192 (1951).

See the paper reviewed above.

Hydrodynamics, Aerodynamics, Acoustics

Truesdell, C. A new definition of a fluid. I. The Stokesian fluid. J. Math. Pures Appl. (9) **29**, 215–244 (1950).

Les formules classiques qui relient les tensions internes $t_i{}^j = -p\delta_i{}^j + \nu_i{}^j$ aux composantes $d_i{}^j$ du tenseur des vitesses de déformations conduisent aux équations de Navier-Stokes; or, celles-ci sont impuissantes à interpréter certains phénomènes dont les gaz à faible pression sont le siège. Dès lors, une révision du concept même du fluide s'impose; et le but de l'auteur est justement de rechercher la forme la plus générale des relations $t_i{}^j = f_i{}^j(d_k{}^l)$ $(i, j, k, l = 1, 2, 3)$ compatibles avec les axiomes de l'analyse dimensionnelle. L'auteur part du concept d'un milieu continu dont l'état est caractérisé par un certain nombre de paramètres (dont les dimensions sont fixées à priori) et de fonctions scalaires; entre ces éléments, l'auteur conçoit axiomatiquement des relations fonctionnelles dont l'analyse dimensionnelle donne la forme. Les échelles des masses M, des longueurs L, du temps T, et de la température θ étant fixées, on admettra l'existence: (1) d'un état de référence naturel (qui sera noté au moyen de l'indice n) et d'une température de référence θ_0; (2) d'un paramètre caractéristique μ de dimensions ML/T, d'une fonction scalaire p; on posera $p_m = -\frac{1}{3}t_i{}^i$. On postulera enfin l'existence des fonctions $f_i{}^j$, développables en séries, telles que:

$$(1) \qquad t_i{}^j = f_i{}^j(\mu_n, \theta_0, p_m, p, \theta, d_k{}^l)$$

se réduisant à $-p\delta_i{}^j$ pour $d_k{}^l = 0$. Si, de plus, le milieu est isotrope, la matrice $t = |t_i{}^j|$ est une fonction matricielle de la matrice $d = |d_k{}^l|$. On déduit alors de (1),

$$(2) \qquad t = -pI + k_1 d + k_2 d^2 + \cdots + k_n d^n + \cdots.$$

Mais un résultat de Reiner [Amer. J. Math. **67**, 350–362 (1945); ces Rev. **7**, 44] donne $d^3 = \mathrm{I}_d d^2 - \mathrm{II}_d d + \mathrm{III}_d I$, où I_d, II_d, III_d sont les trois invariants principaux de $|d_j{}^i|$, et (2) se réduit à $t = \mathfrak{F}_0 I + \mathfrak{F}_1 d + \mathfrak{F}_2 d^2$, les $\mathfrak{F}$ étant des fonctions de θ, θ_0, p, p_m, μ_n, dont il faut maintenant préciser la forme. On trouve, en développant par rapport à μ_n:

$$t_i{}^j = -\delta_i{}^j p + \mu_n(A\,\mathrm{I}_d\delta_i{}^j + Bd_i{}^j) + \frac{\mu_n{}^2}{p}(C\mathrm{I}_d{}^2\delta_i{}^j + D\mathrm{I}_d d_i{}^j + E\mathrm{II}_d\delta_i{}^j + Fd_k{}^l d_l{}^k),$$

$A, B, \cdots, F$ étant des fonctions des arguments sans dimensions θ/θ_0, p/p_n. On retrouve ainsi les formules classiques en négligeant les termes de degré supérieur au premier en μ_n. Mais si la pression est faible, cette approximation peut devenir insuffisante. Dans ce cas, à côté des paramètres de similitude classiques (Reynolds, Mach, etc.), il faut introduire un paramètre nouveau $\mathfrak{Z} = \mu d/p$ (d étant une vitesse de déformation de référence); si ce nombre est faible, les termes en $\mu_n{}^2$ sont négligeables. Sinon, il faut abandonner le concept de Navier et utiliser les formules ci-dessus pour t; l'auteur montre qu'on ne peut plus écrire $p = -\frac{1}{3}t_i{}^i$. Le concept de fluide ci-dessus est distinct de celui de Reiner et Rivlin [Nature **160**, 611 (1947)]; l'auteur illustre la différence sur un exemple. Dans l'élasticité, le tenseur $d_k{}^l$ est remplacé par celui des déformations, dont les éléments ont des dimensions différentes de $d_k{}^l$; il en résulte que l'analyse ci-dessus ne s'applique pas telle quelle au cas des solides élastiques. L'auteur cherche enfin, en vue des applications aux gaz, à définir axiomatiquement le flux thermique. Une analyse analogue à la précédente, l'amène à étendre la relation de Fourier à son modèle de fluide, le coefficient de conductibilité thermique devant être de la forme: $\chi = f(p/p_m, \theta/\theta_0)$. *J. Kravtchenko et R. Gerber.*

Truesdell, Clifford A. T. On Ertel's vorticity theorem. Z. Angew. Math. Physik **2**, 109–114 (1951).

L'auteur donne une interprétation cinématique et diverses extensions du théorème établi par Ertel pour le tourbillon d'un fluide non visqueux. Ce théorème peut prendre la forme suivante: "Soit A une fonction scalaire, constante pour un même élément matériel, et telle que la projection du rotationnel de l'accélération sur les surfaces $A = \text{const.}$ soit nulle; alors: (1) $\alpha|w|\partial A/\partial x_w$ (α densité), où $\partial/\partial x_w$ est la dérivée dans la direction du vecteur tourbillon w, est aussi une quantité constante pour une même particule."

Quand le fluide est visqueux cette expression (1) est variable d'une particule à une autre, mais si le fluide est contenu dans un récipient fini V, il subsiste la propriété:

$$\int_V \alpha|w|\frac{\partial A}{\partial x_w}dV = \text{const.}$$

Divers cas de fluides compressibles non visqueux sont ensuite considérés. Pour terminer l'auteur considère l'écoulement d'un fluide incompressible non homogène (de densité variable dans la masse du fluide). Il montre, que pour une certaine classe de ces mouvements, si le long de l'écoulement la densité tend à s'uniformiser le rotationnel croit.

R. Gerber (Grenoble).

Viguier, Gabriel. Nouvelles équations de la mécanique des fluides visqueux. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **4**, 193–200 (1949). (French. Croatian summary)

After certain vague ruminations concerning the transformations of Green and Stokes, the author writes down a formula, confused by numerous misprints and undefined symbols, supposedly giving the stress in an isotropic viscous fluid when the velocity gradients are "large." The author apparently shares the erroneous opinion of Girault [Essai sur la viscosité en mécanique des fluides, Publ. Sci. Tech. Ministère de l'Air, Paris, no. 4 (1931)] that the coefficients of the terms of second order in the velocity gradients must vanish [cf. the paper reviewed second above, footnote 27.]

C. Truesdell (Bloomington, Ind.).

Delval, J. Le principe de la moindre contrainte appliqué à la dynamique des fluides incompressibles. Acad. Roy. Belgique, Bull. Cl. Sci. (5) **36**, 639–648 (1950).

The author sets up variational principles analogous to Gauss's principle of least constraint and derives from them the dynamical equations for fluids, both inviscid and viscous. Since the author sets up as a side condition div $a=0$, where a is the acceleration, stating erroneously that this constraint follows from the incompressibility of the fluid, his results are valid only for a special and unusual class of flows. He constructs also a variational principle which leads to the Lagrange-Helmholtz vorticity equation for circulation-preserving motions of incompressible fluids, but the result is valid only subject to the condition that the vector product of vorticity by velocity be zero on the boundary or that the boundary be a Bernoullian surface.

C. Truesdell (Bloomington, Ind.).

Magyar, Franz. Geometrie der Wirbelströmung. Anz. Öster. Akad. Wiss. Math.-Nat. Kl. **1949**, 49–52 (1949).

The author suggests that Lamb's extension of Bernoulli's theorem [Proc. London Math. Soc. (1) **9**, 91–92 (1878); Hydrodynamics, 6th ed., Cambridge Univ. Press, 1932, § 165] can be applied to determine hydrodynamic forces.

C. Truesdell (Bloomington, Ind.).

Heinrich, G. Zur Theorie der stationären, reibungsfreien Wirbelströmung. Anz. Öster. Akad. Wiss. Math.-Nat. Kl. **1950**, 76–84 (1950).

This paper is an exposition, without references, of well-known properties of the vorticity field in a flow of an inviscid fluid. The more interesting results were obtained in a more general form by A. Vazsonyi [Quart. Appl. Math. **3**, 29–37 (1945); these Rev. **7**, 226]. *C. Truesdell.*

Schlichting, H., und Scholz, N. Über die theoretische Berechnung der Strömungsverluste eines ebenen Schaufelgitters. Ing.-Arch. **19**, 42–65 (1951).

This is an attempt to set up a practical procedure for prediction of losses in incompressible flow through cascades. For the pressure distribution on the blades, an extended thin-airfoil method is proposed, as used by several authors. Here it is limited to "shock-free entry." Boundary-layer calculations are made by the methods of Pohlhausen or Gruschwitz, for laminar or turbulent layers, respectively. These calculations are analogous to those for single airfoils (but in cascades there is much greater uncertainty about the location of transition because of the low Reynolds numbers and high turbulence). A careful discussion is given of the meaning and origins of various "losses" in cascade flows. Numerical examples are carried out which clearly show the recognized differences between accelerating (e.g. turbine) and decelerating (e.g. compressor) cascades.

W. R. Sears (Ithaca, N. Y.).

Lieblein, V. Zur Berechnung der Auftriebscharakteristik eines Profils im Gitterverband. Ing.-Arch. **18**, 281–290 (1950).

This paper is concerned with the calculation of the lift of thick airfoils of a lattice in two-dimensional incompressible flow by an extension of the Birnbaum-Glauert method given by E. Pistolesi [Aerotecnica **17**, 484–506 (1937)] and by Pistolesi and Toniolo [ibid. **18**, 1065–1094 (1938)]. According to the editor "a further work of Pistolesi and Toniolo which contains considerations similar to these was obviously not known to the author." *C. Saltzer.*

Libby, Paul A., and Reiss, Howard R. The design of two-dimensional contraction sections. Quart. Appl. Math. **9**, 95–98 (1951).

Contraction sections of infinite length are determined such that for an incompressible, potential flow the speed will be everywhere monotonically increasing in the direction of flow. This is accomplished by means of the particular complex potential $dw/dz=c \ln[(w-b)/(w-a)]$, where w is the complex velocity. Values of the constants are determined for which the condition of increasing speed is satisfied. For other values the speed is shown to be locally decreasing.

P. W. Ketchum (Urbana, Ill.).

Escande, L. Calcul des oscillations de l'eau dans des bassins communicants. Srpska Akad. Nauka. Zbornik Radova, Knj. **6**. Hidrotehnički Institut, Knj. **1**, 9–36 (1951). (Serbo-Croatian. French summary)

Jacobsen, Lydik S. Impulsive hydrodynamics of fluid inside a cylindrical tank and of fluid surrounding a cylindrical pier. Bull. Seismol. Soc. America **39**, 189–204 (1949).

The forces exerted by the water upon a partly filled cylindrical tank and upon a cylindrical pier surrounded by water in case of an earthquake are determined under the following assumptions. (1) The tanks and piers experience translatory impulsive ground displacements in the horizontal direction only. (2) The circular cylindrical boundaries behave as rigid surfaces. (3) The fluid is incompressible and inviscid. The investigation is restricted to a brief impulsive time interval; hence the convective part of the acceleration is neglected in the Euler equations. The solution for the velocity potential, and hydrodynamic pressure distribution is given in the form of a series of Bessel functions. Their asymptotic properties are used for evaluation of effective hydrodynamic masses and moments. Good agreement with experiments is shown. For certain examples the velocity field and the pressure distribution upon the cylindric surface are also evaluated and represented graphically. This investigation is probably the first major contribution to a new branch of fluid dynamics which may be called "seismo-hydrodynamics." *P. Németi* (Washington, D. C.).

Weitz, Mortimer, and Keller, Joseph B. Reflection of water waves from floating ice in water of finite depth. Comm. Pure Appl. Math. **3**, 305–318 (1950).

Mathematically, the problem of the titic reduces to the solution of the following boundary value problem in the strip $-\infty<x<\infty$, $0\leqq y\leqq a$. It is required to solve the partial differential equation $\phi_{xx}+\phi_{yy}-k^2\phi=0$ (k real) subject to the boundary conditions $\phi_y=0$ for $-\infty<x<\infty$, $y=0$, $\phi_y=\alpha\phi$ for $x<0$, $y=a$, $\phi_y=\beta\phi$ for $x>0$, $y=a$, with certain order conditions for $|x|\to\infty$. For $\alpha=0$, $\beta>0$, this problem has been treated by the reviewer [Amer. J. Math. **70**, 730–748 (1948); these Rev. **10**, 490]. The present problem is formulated as an integral equation of the convolution type for $\phi(x, a)$ and solved by complex Fourier transform methods. The analysis appears to suggest that this problem may be formulated as an integral equation of the Wiener-Hopf type. *A. E. Heins* (Pittsburgh, Pa.).

Haskind, M. D. The pressure of waves on a barrier. Akad. Nauk SSSR. Inženernyĭ Sbornik **4**, no. 2, 147–160 (1948). (Russian)

The author considers the problem of the effect of a long plane barrier immersed a finite depth into infinitely deep water with harmonic surface waves. The problem is solved

using the linearized theory of surface waves. The author obtains the amplitudes of the reflected and transmitted waves, the amplitudes of the force and moment on the barrier resulting from the waves, the phase shift in these quantities, and finally the mean force and moment over a period. This problem was also considered at about the same time by F. Ursell [Proc. Cambridge Philos. Soc. **43**, 374–382 ((1947); these Rev. **9**, 117 (see this review for a statement of the mathematical problem involved)]. Although the methods are different, the results seem to be in substantial agreement where they overlap. The author concludes with a short discussion of the effect of finite length of the barrier. *J. V. Wehausen* (Providence, R. I.).

Longuet-Higgins, M. S. A theory of the origin of microseisms. Philos. Trans. Roy. Soc. London. Ser. A. **243**, 1–35 (1950).

The author, using the Lagrangean form of the equations of motion, analyses the pressure variation $p(x, z, t)$, and especially the mean pressure variation $\bar{p}(z, t)$ over a wavelength λ, for a periodic two-dimensional wave on water of either infinite or finite depth. He obtains, without any assumptions regarding the smallness of the wave amplitude or slope, the expression

$$\frac{\bar{p}-p_s}{\rho}=\frac{1}{\lambda}\frac{\partial^2}{\partial t^2}\int_0^\lambda \tfrac{1}{2}\zeta^2 dx-\frac{1}{\lambda}\int_0^\lambda w^2 dx,$$

where p_s is the constant free-surface pressure, ρ the density, ζ the equation of the free surface, and w the z-component (z measured downwards from the mean free surface) of the motion. The formula is applied to the special case of two progressive sinusoidal waves of common frequency σ, wavelength λ, and amplitudes a_1, a_2 travelling in opposite directions on water of depth h to obtain, to the 2d order of approximation, $(\bar{p}-p_s)/\rho-gh=-2a_1a_2\sigma^2\cos 2\sigma t$. Thus, when this order of approximation is kept, there is still a mean pressure fluctuation on the bottom (if $a_1a_2\neq 0$) which is independent of the depth [cf. R. Miche, Ann. Ponts Chaussées **1944** (114^e année), 25–78 (1944), especially p. 73; these Rev. **7**, 348]. This analysis is extended by a 2-dimensional Fourier integral analysis to more general types of wave motion [but, it seems to the reviewer, not as general as the author states; however, this does not affect the geophysical applicability of the results].

Since the author, in retaining higher order terms, is dealing with disturbances which are propagated to the bottom of a deep ocean (i.e. $h\sim c/\sigma$, c the velocity of sound), it is necessary to consider the effect of the compressibility of the water on the wave motion. The wave theory for a heavy compressible (equation of state: $p-p_s=c^2(\rho-\rho_s)$) fluid is carried through to the second order of approximation for motion which in the first approximation is the resultant of two progressive waves of the same wave-length travelling in opposite directions. He finds that near the surface (roughly, $z<\lambda_s$ = length of a gravity wave) the motion is chiefly like that of a gravity wave in an incompressible fluid, but that lower down ($z\sim\lambda_s$ = length of the corresponding compression wave) the motion is small and due chiefly to the compressibility.

An elastic sea bed with an overlying ocean of depth h is next assumed. After considering the displacement of the sea bed by a concentrated force on the surface, the author uses the Fourier analysis developed earlier to estimate the effect of a disturbed ocean area of finite extent on the sea bed. From the geophysical point of view this is of course the real aim of the paper. When the wave motion in a region of the ocean is of the right sort (approximately standing waves) the unattenuated pressure fluctuation on the bottom, as predicted by the second-order theory, can cause microseismic waves in the sea bed which are observable far inland. *J. V. Wehausen* (Providence, R. I.).

Binnie, A. M. The theory of waves travelling on the core in a swirling liquid. Proc. Roy. Soc. London. Ser. A. **205**, 530–540 (1951).

When a swirling liquid passes through a pipe or nozzle, a hollow core is formed provided that the swirl is sufficiently large. The author examines waves moving under centrifugal force and surface tension along the core in a swirling liquid. The waves may be of varicose form in which the cross-section of the core remains circular, or they may be helical, giving the core the shape of a multithreaded screw. The relation is obtained between the lengths of the waves and their axial and angular velocities; at a critical length the waves have a minimum velocity. The group velocities are determined and are shown to be negative in certain conditions. It is found that waves can exist which move so slowly that they should be readily visible although the core may be revolving at high speed. *L. M. Milne-Thomson.*

Fišman, I. M. On the motion of a strongly viscous liquid between a journal and a bearing. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 593–610 (1950). (Russian)

Ce problème a été étudié par différents auteurs dont N. Joukovsky, S. Tchapliguine et A. Sommerfeld qui l'ont résolu pour le cas stationnaire. W. Harrison a trouvé une solution approchée pour le cas non stationnaire lorsque le jeu (δ) entre le tourillon et le coussinet est très petit. Dans toutes ces recherches on a négligé les forces d'inertie du liquide. L'auteur étudie le cas non stationnaire en introduisant les forces d'inertie. On utilise l'équation de Navier-Stokes en coordonnées polaires pour un mouvement plan d'un liquide incompressible, en supposant le jeu (δ) très petit. En introduisant la fonction de courant, on obtient en première approximation une équation biharmonique, en négligeant dans le second membre les forces d'inertie. En deuxième approximation on introduit les termes d'inertie et l'on obtient une solution approchée de la fonction de courant pour un δ très petit. L'auteur étudie après le mouvement du tourillon pour une charge constante. Il résoud le système en utilisant la méthode de Kryloff et Bogoliouboff et montre que la solution de Harrison n'est pas tout à fait correcte; il démontre la stabilité de la solution de Joukovsky et Tchapliguine. L'auteur considère aussi le cas d'une charge périodique. *M. Kiveliovitch* (Paris).

Chien, Wei-Zang. Hydrodynamic theory of lubrication for plane sliders of finite width. Chinese J. Phys. **7**, 278–299 (1949). (English. Chinese summary)

The incompressible viscous fluid motion associated with thin film lubrication is studied. The solution (velocity and pressure) of the Navier-Stokes equations is represented as a power series in the film thickness coordinates. The zeroth order problem arising from this expansion is then treated by a perturbation in the thickness itself. The lowest order perturbation appears as the solution of the classical Reynolds' problem. This problem is formulated as a variational problem whose direct solution, in certain examples, checks well with a more precise integration of the Reynolds' equations. *G. F. Carrier* (Providence, R. I.).

Wang, Tsu-Tang. Theory of hydrodynamic lubrication of plane slider and full journal bearing with side leakage. Eng. Rep. Nat. Tsing Hua Univ. **4**, no. 2, 96–106 (1950). (Chinese)

The author has calculated approximately the pressure and skin-friction coefficients for the cases of plane slider and full-journal bearings with side leakage by the Rayleigh-Ritz Method. Previously, W. Z. Chien [see the preceding review] has shown that in the theory of lubrication the solution of Reynolds' equation can be replaced by a variational problem.

Y. H. Kuo (Ithaca, N. Y.).

Archibald, F. R. A simple hydrodynamic thrust bearing. Trans. A.S.M.E. **72**, 393–400 (1950).

The analysis of a slider bearing with a stepped oil film is carried out, taking into account the side leakage. The bearings considered are of rectangular and sectorial shape. The results are compared with the classical (no side leakage) results.

G. F. Carrier (Providence, R. I.).

***Polubarinova-Kochina, P. Ya., and Falkovich, S. B. Theory of filtration of liquids in porous media.** Advances in Applied Mechanics, vol. 2, edited by Richard von Mises and Theodore von Kármán, pp. 153–225. Academic Press, Inc., New York, N. Y., 1951. $6.50.

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 629–674 (1947); these Rev. **10**, 73.

Finston, M. Thermal effects in calendering viscous fluids. J. Appl. Mech. **18**, 12–18 (1951).

The equations implying conservation of mass, momentum and energy are written in bipolar coordinates for an incompressible fluid whose viscosity varies linearly with temperature. Since, during calendering, the distance between rolls is small compared to the distance between inlet and outlet sections, these equations can be replaced by much simpler approximations, and the resulting system solved by perturbation about the constant-viscosity solution. This yields sets of linear partial differential equations for the velocity, pressure and temperature distributions, which are solved. Solution of an example is carried out, affording an explanation of blistering observed during calendering of linoleum.

R. E. Gaskell (Seattle, Wash.).

Kronig, R., and Bruijsten, J. On the theory of the heat and mass transfer from a sphere in a flowing medium at low values of Reynolds' number. Appl. Sci. Research A. **2**, 439–446 (1951).

Analysis is restricted to forced convection due to laminar flow around a sphere without viscous energy dissipation. The solution of the basic convection heat conduction equation is approached by a successive approximation method assuming the flow Peclet number, Pe, to be small. It is shown that the zero approximation is more advantageously taken as convection from a point source in steady flow rather than conduction from a sphere in a stationary fluid. On this basis, averaging over the surface of the sphere, it is shown that the Nusselt number is

$$\mathrm{Nu}=2+\tfrac{1}{2}\mathrm{Pe}+\frac{581}{1920}\mathrm{Pe}^2+\cdots.$$

The analytical details give consideration to the angular as well as radial temperature distribution.

N. A. Hall.

Kawaguti, Mitutosi. Numerical solution for the viscous flow past a sphere. Rep. Inst. Sci. Tech. Univ. Tokyo **4**, 154–158 (1950). (Japanese. English summary)

"The viscous flow past a sphere is studied by use of a numerical method, which is similar to that used by Thom [Proc. Roy. Soc. London. Ser. A. **141**, 651–669 (1933)] for studying the viscous flow past a circular cylinder. The case $R=20$ is dealt with, where $R=Ud/\nu$ is the Reynolds number, U being the velocity of the oncoming stream, d the diameter of the sphere, and ν the kinematic viscosity."

From the author's summary.

Shvets, M. E. Method of successive approximations for the solution of certain problems in aerodynamics. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1286**, 20 pp. (1951).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 257–266 (1949); these Rev. **11**, 277.

Nigam, Swami Dayal. Rotation of an infinite plane lamina: Boundary layer growth: Motion started impulsively from rest. Quart. Appl. Math. **9**, 89–91 (1951).

Cette note est consacrée à l'étude du mouvement commençant, engendré dans un fluide visqueux au repos par la mise brusque en rotation d'un plan infini autour d'un axe perpendiculaire. En considérant le temps comme infiniment petit, et en négligeant les termes du deuxième ordre, l'auteur ramène le problème à la résolution d'un système d'équations différentielles linéaires du second ordre. Il en donne la solution explicite et construit les courbes représentatives.

R. Gerber (Grenoble).

Lock, R. C. The velocity distribution in the laminar boundary layer between parallel streams. Quart. J. Mech. Appl. Math. **4**, 42–63 (1951).

The author considers the laminar boundary layer between two parallel streams of homogeneous incompressible fluids of different density and viscosity. The usual method of treating laminar boundary layers is applied, but the discussion of the boundary conditions is more complicated. The approximate solution of the problem has been carried out by G. I. Taylor, in an unpublished note, by using the von Kármán momentum integral method, but the author also carries out the detailed solution by solving differential equations of the Blasius type by a method similar to that used by Lessen [Tech. Rep. Nat. Adv. Comm. Aeronaut. no. **979** (1950); these Rev. **12**, 648] for two streams of the same fluid. It is shown that the solutions depend only on the ratio U_2/U_1 of the velocities of the two streams and on the product $\rho\mu$ of the corresponding density and viscosity ratios.

C. C. Lin (Cambridge, Mass.).

Borodin, V. A., and Dityakin, Y. F. Unstable capillary waves on surface of separation of two viscous fluids. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1281**, 19 pp. (1951).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 267–276 (1949); these Rev. **11**, 66.

Hölder, Ernst. Klassische und relativistische Gasdynamik als Variationsproblem. Math. Nachr. **4**, 366–381 (1951).

The author derives a variational principle of the form

$$\delta\int\int f(x, y, z, p, q)dxdy=0$$

where $s=\psi(x, y)$, $p=\partial\psi/\partial x$, $q=\partial\psi/\partial y$, for various special types of adiabatic flows. The method used to determine the function f involves the use of conjugate momenta variables and the identification of these with quantities conserved under the flow. The last part of the paper contains a discussion of some special flows in relativistic hydrodynamics. A variational principle for arbitrary flows, including ones with shocks, which is of a different form from that given above has been derived by the reviewer [Proc. Symposia Appl. Math., vol. I, pp. 148–157, Amer. Math. Soc., New York, 1949; these Rev. 11, 222]. *A. H. Taub.*

Wang, Chi-Teh. A note on Bateman's variational principle for compressible fluid flow. Quart. Appl. Math. **9**, 99–102 (1951).

Le principe variationnel établi par Bateman pour le mouvement permanent et irrotationnel d'un fluide compressible n'est plus valable quand le fluide s'étend à l'infini. L'auteur montre comment ce principe doit être modifié dans ce cas. Il utilise pour celà la décomposition du potentiel des vitesses, suggérée par la méthode approchée de Rayleigh-Ritz, en une somme de deux termes, dont l'un représente le potentiel des vitesses du fluide incompressible correspondant. *R. Gerber* (Grenoble).

Sedov, L. I. On the general theory of plane-parallel gas flows. Vestnik Moskov. Univ. **4**, no. 9, 3–14 (1949). (Russian)

Formal discussion of the linearization of the equations of a steady barotropic potential gas flow in two dimensions for an arbitrary pressure density relation $\rho=f(p)$ by means of either the Mollenbroek-Chaplygin or Legendre transformations. In particular, the author indicates functions $f(p)$ which approximate the adiabatic relation and lead to equations for the stream-function which admit particular solutions involving Bessel functions. *L. Bers.*

Munk, Max M. The Rankine gas flow in the hodograph plane. Symposium on theoretical compressible flow, 28 June 1949. Naval Ordnance Laboratory, White Oak, Md., Rep. **NOLR-1132**, pp. 43–51 (1950).

The Rankine flow in the plane potential flow of a perfect incompressible fluid is generated by the superposition of the flow from one point source upon a parallel flow. The author indicates how the stream function for the corresponding flow of a perfect gas may be obtained in the hodograph plane as an infinite series. The flow in the physical plane is then obtained by taking the gradient of the Legendre potential function L. *M. H. Martin* (College Park, Md.).

Gelbart, Abe, and Resch, Daniel. A method of computing subsonic flows around given airfoils. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2057**, 34 pp. (1950).

In a previous paper [A. Gelbart, same Tech. Notes, no. 1170 (1947); these Rev. **8**, 417] the senior author derived the following formula for transforming an incompressible flow past a circle in the ζ-plane given by the complex potential $G(\zeta)=q(\zeta+R^2/\zeta)-(i\Gamma/2\pi)\log(\zeta/R)$ into a flow past a profile P in the z-plane satisfying the Chaplygin simplified pressure-volume relation ("$\gamma=-1$"):

$$(1)\qquad z=z(\zeta)=f(\zeta)-\frac{1}{4}\int\frac{G'(\zeta)^2d\zeta}{f'(\zeta)}.$$

Here $f'(\zeta)=\sum_0^\infty b_n\zeta^{-n}$ is an arbitrary analytic function. The coefficient b_0 determines the speed at infinity, the coefficient b_1 is determined by the condition that (1) map $|\zeta|=1$ onto a closed curve. The coefficients $b_2, b_3, \cdots$ determine the shape of P. [Cf. C. C. Lin, Quart. Appl. Math. **4**, 291–297 (1946); P. Germain, C. R. Acad. Sci. Paris **223**, 532–534 (1946); these Rev. **8**, 418, 237.] In the present paper the authors describe a simple method of choosing the coefficients b_n in such a way that P should approximate closely, though not arbitrarily closely, a given profile P_0. The method consists in minimizing the maximum of $|z(Re^{i\theta})-\omega(Re^{i\theta})|$, where $\omega(\zeta)$ is the function mapping $|\zeta|>R$ conformally onto the domain exterior to P_0. Numerical computations are presented for the case when P_0 is a symmetrical Joukowski profile, for various values of the angle of attack and the free stream Mach number. *L. Bers* (Los Angeles, Calif.).

Costello, George R. Method of designing cascade blades with prescribed velocity distributions in compressible potential flows. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **978**, 11 pp. (1950).

A procedure for the design of cascade blades in an infinite straight lattice with prescribed velocity distributions in a subsonic compressible flow subject to the linear pressure-volume relation is formulated based on a method given by C. C. Lin [Quart. Appl. Math. **4**, 291–297 (1946); J. Math. Physics **28**, 117–130 (1949); these Rev. **8**, 418; **11**, 64] and A. Gelbart [Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1170 (1947); these Rev. **8**, 417]. A modification of this procedure which treats the analogous problem for an isolated airfoil is also given. *C. Saltzer* (Cleveland, Ohio).

Costello, George R., Cummings, Robert L., and Sinnette, John T., Jr. Detailed computational procedure for design of cascade blades with prescribed velocity distributions in compressible potential flows. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2281**, 49 pp. (1951).

A detailed outline for the computations involved in the above procedure for the design of cascade blades is given together with tables to be used in the numerical evaluation of integrals occurring in the calculations. Four examples are included. In addition, the linear pressure-volume compressible flow ($\gamma=-1$) is interpreted as corresponding to the usual compressible flow ($\gamma=1.4$) under the assumption that the magnitudes of the velocities of the flows are proportional and the parameters of the linear pressure-volume flow are calculated from the parameters of the given flow using this assumption. *C. Saltzer* (Cleveland, Ohio).

Bugaenko, G. A. On the problem of gas flow over an infinite cascade using Chaplygin's approximation. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1298**, 16 pp. (1951).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 449–456 (1949); these Rev. **11**, 225.

Gruschwitz, E. Calcul approché de la couche limite laminaire en écoulement compressible sur une paroi non conductrice de la chaleur. Application numérique et discussion par E. A. Eichelbrenner. O.N.E.R.A. Publ. no. **47**, vii+38 pp. (1950).

The author develops an extension of the Kármán-Pohlhausen method to the case of the compressible fluid flowing over an insulated boundary. Integrated relations are obtained for both the momentum and the energy equations. Detailed calculations are given for Prandtl number equal to unity and 0.725, with the assumption that viscosity is proportional to the absolute temperature. The polynomial used

for the representation of the velocity distribution is of the fourth degree, the same as that used by other authors for the case of compressible fluid. *C. C. Lin.*

Chapman, Dean R. Laminar mixing of a compressible fluid. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. 958, 7 pp. (1950).

A theoretical investigation of the velocity profiles for laminar mixing of a high-velocity stream with a region of fluid at rest has been made assuming that the Prandtl number is unity. The method is essentially the one used by von Kármán and Tsien for the boundary layer over a flat plate. Detailed calculations have been made for free stream Mach numbers of 0, 1, 2, 3 and 5. The viscosity is taken to vary either with the 0.76 power or first power of absolute temperature.

C. C. Lin (Cambridge, Mass.).

Young, A. D. The equations of motion and energy and the velocity profile of a turbulent boundary layer in a compressible fluid. Coll. Aeronaut. Cranfield. Rep. no. 42, 14 pp. (2 plates) (1951).

The author derives the equations of motion and energy for a turbulent boundary layer in a compressible fluid and makes some deductions on the basis of the mixture length theory and other classical concepts. One conclusion is that for Mach numbers of the order of 2.5 or less and for a wide range of heat transfer conditions, the form of the velocity profile in the turbulent boundary layer will differ very little from that for an incompressible fluid and the same Reynolds number, in agreement with existing experimental results.

C. C. Lin (Cambridge, Mass.).

Spreiter, John R. Similarity laws for transonic flow about wings of finite span. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 2273, 24 pp. (1951).

The method previously employed by von Kármán [J. Math. Phys. 26, 182–190 (1947); these Rev. 9, 217] to derive a transonic similarity law for two-dimensional flow is here extended to study three-dimensional flow about wings of finite span. It is shown for transonic flows of free-stream Mach number M_0 that similarity depends upon two parameters, $(1-M_0^2)^{\frac{1}{3}}/[(\gamma+1)\tau]^{\frac{1}{3}}$ and $(1-M_0^2)^{\frac{1}{3}}A$, where τ is a wing-thickness parameter and A is the aspect ratio (γ is the ratio of specific heats, taken here to be 1.4). The first of these parameters appeared in von Kármán's similarity law. Similarity laws are also derived here for linearized subsonic and supersonic wing theory. It is noted that one of the forms of these laws coincides with the transonic similarity relations described above. *E. N. Nilson.*

Kuo, Yung-Huai. On the stability of two-dimensional smooth transonic flows. J. Aeronaut. Sci. 18, 1–6, 54 (1951).

The author studies the stability of smooth transonic flows with respect to theoretically possible disturbances. The following conclusions are drawn: (1) smooth decelerating transonic flows over bodies at negative curvature are unstable to compression pulses; (2) smooth accelerating transonic flows over any surface are stable; and (3) smooth transonic flows with local supersonic region are unstable. Some of these conclusions are modified by later investigations of the author.

C. C. Lin (Cambridge, Mass.).

Oswatitsch, Klaus. Die Geschwindigkeitsverteilung an symmetrischen Profilen beim Auftreten lokaler Überschallgebiete. Acta Physica Austriaca 4, 228–271 (1950).

Amplification of an earlier paper [Z. Angew. Math. Mech. 30, 17–24 (1950); these Rev. 11, 555].

J. H. Giese (Havre de Grace, Md.).

Cherry, T. M. Relation between Bergman's and Chaplygin's methods of solving the hodograph equation. Quart. Appl. Math. 9, 92–94 (1951).

Let $\tau^{\frac{1}{2}}\cos\theta$, $\tau^{\frac{1}{2}}\sin\theta$ be the velocity components of a steady potential adiabatic gas flow. The stream function ψ considered as a function of τ and θ satisfies a linear partial differential equation given by S. A. Chaplygin. Chaplygin [Učenye Zapiski Imp. Moskov. Univ., Otd. Fiz.-Mat. 21, 1–121 (1904)] writes the "general solution" in the form $\sum c_\nu\psi_\nu(\tau)e^{i\nu\theta}$ where the ν's are real numbers ($\nu\neq -2, -3, \cdots$), $\psi_\nu(\tau)$ certain hypergeometric functions, and c_ν arbitrary constants. On the other hand, Bergman's operator method permits one to write down a "general solution" of Chaplygin's equation in a form involving an arbitrary analytic function of a complex variable [S. Bergman, Tech. Notes Nat. Adv. Comm. Aeronaut., no. 972 (1945); R. von Mises and M. Schiffer, Advances in Applied Mechanics, vol. 1, pp. 249–285, Academic Press, New York, 1948; these Rev. 7, 342; 10, 642]. Using his own [Proc. Roy. Soc. London. Ser. A. 192, 45–72 (1947); these Rev. 9, 544] and Lighthill's [ibid. 191, 323–341, 341–351, 352–369 (1947); these Rev. 9, 391, 351, 391] results on the Chaplygin solution the author rewrites it in Bergman's form. *L. Bers.*

Bruniak, R. Über eine Anwendung des Croccoschen Wirbelsatzes. Österreich. Ing.-Arch. 4, 325–334 (1950).

The gradients of the energy E and entropy S in Crocco's vortex theorem $\mathfrak{v}\times\mathfrak{w}=\nabla E-T\nabla S$ are calculated from the formulae for an oblique shock and applied to a shock curve. The nonlinear partial differential equation for the stream function is calculated for a special case of flow around a profile. *M. H. Martin* (College Park, Md.).

Oswatitsch, K., und Bruniak, R. Über eine Anwendung des Croccoschen Wirbelsatzes. Österreich. Ing.-Arch. 5, 209 (1951).

Remarks by Oswatitsch and answer by Bruniak on the paper reviewed above.

Coburn, N. Compressible supersonic flow in jets under the Kármán-Tsien pressure-volume relation. J. Appl. Phys. 22, 124–130 (1951).

The jet problem considered by S. A. Chaplygin [Učenye Zapiski Imp. Moskov. Univ., Otd. Fiz.-Mat. 21, 1–121 (1904); for a translation see Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1063 (1944); these Rev. 7, 495] for subsonic flow and by F. Frankl [Izvestiya Akad. Nauk SSSR 9, 121–143 (1945); these Rev. 7, 496] for transonic flow of a polytropic gas is treated for the Kármán-Tsien gas. This simplification enables the author to study the mapping from the hodograph to the physical plane in detail and the periodicity in the jet. *M. H. Martin.*

Johannesen, N. H., and Meyer, R. E. Axially-symmetrical supersonic flow near the centre of an expansion. Aeronaut. Quart. 2, 127–142 (1950).

The Prandtl-Meyer flow for steady, plane, irrotational supersonic flow of a polytropic gas around a corner is extended to axially symmetric flow. The flow quantities are

assumed constant on each member of a one-parameter family of right circular cones with a common base. The normal to the base at its center is the axis of symmetry. The angle φ between the base and the lateral surface of the cone is the parameter of the family. The components u, v of the velocity along and perpendicular to the lateral surface of the cone are expanded into series:

$$u=u_0(\varphi)+Ru_1(\varphi)+\cdots, \quad v=v_0(\varphi)+Rv_1(\varphi)+\cdots,$$

where R is the distance along a generator of the cone measured from the base of the cone, and substituted into the equation of motion. The first two terms in each of these series are determined and two solutions are obtained. One approaches a simple wave as $R\to 0$ and the other a uniform flow. The results are applied to a jet from a nozzle discharging into a container in which the pressure is lower than in the nozzle to obtain a formula for the curvature of the boundary of the jet at the lip of the nozzle. The nature of the flow near an edge in the boundary wall is also investigated.

M. H. Martin (College Park, Md.).

Cabannes, Henri. Étude de la courbure au sommet de l'onde de choc attachée dans un écoulement de révolution. C. R. Acad. Sci. Paris **232**, 481–483 (1951).

In the axisymmetrical supersonic flow past a pointed projectile of revolution with attached shock, the ratio of the shock radius of curvature to the projectile radius of curvature at the nose is studied on an exact theory. In the range of Mach numbers for which part (at least) of the flow behind the shock is subsonic (although the shock has been chosen as the weaker of the two solutions for a given head semi-angle) there is a "singular" Mach number at which the said ratio passes through zero. This recalls the similar behaviour in plane flow, which Crocco [Atti del Primo Congresso dell'Unione Matematica Italiana, pp. 597–615, Bologna, 1937] considered might correspond to the Mach number for shock detachment. *M. J. Lighthill* (Manchester).

Ovsiannikov, L. V. Gas flow with straight transition line. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1295**, 13 pp. (1951).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 537–542 (1949); these Rev. **11**, 271.

Sakurai, Akira. On the thickness of plane shock waves in a gas in turbulent motion. J. Phys. Soc. Japan **5**, 114–117 (1950).

The author attempts to develop a theory to explain the thickness of the shock in terms of turbulent motion. While the basic ideas are sound, much simplification has to be made to carry out the details, as is generally the case with turbulence studies in a compressible fluid. *C. C. Lin.*

Brown, Clinton E. The reversibility theorem for thin airfoils in subsonic and supersonic flow. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **986**, 3 pp. (1950).

Cf. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1944** (1949); these Rev. **11**, 753.

Lomax, Harvard, Heaslet, Max. A., and Fuller, Franklyn B. Formulas for source, doublet, and vortex distributions in supersonic wing theory. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2252**, 35 pp. (1950).

Particular attention is given to the difficulties introduced by singularities arising in various improper integrals. It would be of interest to compare the authors' approximate treatment of the low aspect ratio rectangular wing with that of K. Stewartson [Proc. Cambridge Philos. Soc. **46**, 307–315 (1950); these Rev. **11**, 699]. *J. W. Miles* (Auckland).

Lomax, Harvard, Heaslet, Max. A., and Fuller, Franklyn B. Three-dimensional, unsteady-lift problems in high-speed flight. Basic concepts. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2256**, 39 pp. (1950).

Classical methods of treating the wave equation are discussed and several well-known results derived as illustrative applications. The validity of strip theory for simple plan forms with straight trailing edges has been proved more generally by the reviewer [J. Aeronaut. Sci. **17**, 127 (1950)]. The authors' rather cursory treatment of "slender wing" theory for unsteady flow appears to overlook the fact that, for the fixed co-ordinates selected, the regions delineated by the mixed boundary conditions of continuity of pressure and tangency of flow are time-dependent.

J. W. Miles (Auckland).

Krasilschikova, E. A. Disturbed motion of air caused by vibrations of a wing moving at supersonic speed. Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio, Tech. Rep. no. **102-AC49/4-34 (GDAM A9-T-24)** 39 pp. (1949).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 147–164 (1947); these Rev. **9**, 392.

Karpovich, E. A., and Frankl, F. I. Resistance of a delta wing in a supersonic flow. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1283**, 6 pp. (1951).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 495–496 (1947); these Rev. **9**, 162.

Struminsky, V. V. Sideslip in a viscous compressible gas. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1276**, 8 pp. (1951).

Translated from Doklady Akad. Nauk SSSR N. S. **54**, 769–772 (1946); cf. these Rev. **8**, 610.

Broer, L. J. F. On the influence of acoustic relaxation on compressible flow. Appl. Sci. Research A. **2**, 447–468 (1951).

The chief object of this paper is to discover perturbations in the motion of a gas produced by heat capacity lag, in distinction to previous work [by the reviewer, J. Chem. Phys. **14**, 150–164 (1946)] in which the unperturbed flow is used as a basis for calculation of the entropy increases only. It is shown that even in an inviscid fluid, vortices can appear due to heat capacity lag. For example, the flow at the nose of an impact tube is shown to shed ring vortices encircling the tube.

Turning to two-dimensional supersonic flow and one-dimensional unsteady flow the author conventionally derives the equations of the characteristics and shows that these correspond to the propagation of waves with sonic velocity corresponding to the high frequency (large) speed of sound. However, this higher sonic velocity is only the velocity of propagation of "precursor waves" in the sense of Sommerfeld and Brillouin, and in the main disturbance which follows adjustments in the lagging heat capacities are made. In connection with this type of disturbance the author discusses shock waves and finds a minimum shock strength possible. He finds that a shock wave is made up of a discontinuity followed by a region in which the lagging heat

capacity is adjusted. Thus the shock waves have a total thickness which depends upon the relaxation time of the gas.

The remainder of this paper is devoted to the discussion of "transonic flow" by which he means steady flow at velocities intermediate between the low- and high-frequency sonic speeds. He gives two arguments to show that flow in this velocity range is unstable. In the first of these he shows that in strictly one-dimensional "transonic" flow a velocity perturbation inevitably involves a velocity gradient tending to increase this perturbation as one moves in the flow-direction. This argument is inconclusive to the reviewer since in a real steady flow a perturbation in velocity would be accompanied by variations in stream tube area, which would destroy the validity of his analysis. His analysis is applicable only to the case of thickened shock waves, which are strictly one-dimensional. It means then that if there is a velocity perturbation in the "transonic" region of a strictly one-dimensional flow, it must constitute a portion of a shock wave and does not correspond to any real flow which is not strictly one-dimensional.

In his second argument he shows that entropy increases inevitably result from velocity perturbations in the "transonic" region. He interprets this as another indication that these velocity perturbations will increase. The reviewer's interpretation is that the existence of such entropy increases, is merely further evidence that he is in the midst of one of his thickened shock waves. More generally it can be seen that the statement that uniform steady flow in a certain velocity range is unstable has no meaning since the flow velocity depends upon the coordinate system used. Any such instability must be related to gradients in the unperturbed flow rather than to the flow velocity. [Courtesy of Applied Mechanics Reviews.] *A. R. Kantrowitz.*

Eggers, A. J., Jr. One-dimensional flows of an imperfect diatomic gas. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. 959, 11 pp. (1950).

Even when the effects of heat-capacity lag, and viscosity and heat conduction are neglected, there are deviations from perfect gas flow due to small inaccuracies in the equation of state of a perfect gas and the slow variations of the specific heats with temperature and pressure. The author uses Berthelot's equation of states to describe the equilibrium states of the gas. Then only the temperature-dependent part of the specific heats needs to be specified. For aerodynamic applications, where diatomic gases are involved, the number of translational and rotational degrees of freedom is constant, and only the variation with temperature of the vibrational heat capacity need be considered. This is done through the well-known Einstein function involving the characteristic temperature θ. One-dimensional isentropic flow equations and shock relations are then obtained. Since the deviations from the perfect gas flows are expected to be small, the author then gives approximate relations which are correction formulas for gaseous imperfections. *H. S. Tsien.*

Roy, Maurice. Notions fondamentales d'aérothermodynamique. O.N.E.R.A. Publ. no. 44, iii+34 pp. (1950).

This exposition of the thermodynamics of fluids is intended for engineers. A fuller treatment is in the author's book [Mécanique des milieux continus et déformables, Gauthier-Villars, Paris, 1950; these Rev. 12, 550]. Here there is greater emphasis on applications: a motor in rapid translation, propulsion by reaction, efficiency of a screw propeller, etc. *C. Truesdell* (Bloomington, Ind.).

Hassanein, S. A. M. A wave pattern changing with time as a solution of the vorticity equation for non-divergent motion. Tellus 1, no. 4, 58–60 (1949).

The author studies the vorticity equation obtained by Rossby [J. Marine Research 2, 38–55 (1939)] to describe the horizontal motion of the earth's atmosphere, assumed to be a homogeneous inviscid incompressible fluid. He obtains a special solution which includes as a special case others given by Rossby [loc. cit.], Haurwitz [ibid. 3, 35–50 (1940)], and Machta [J. Meteorol. 6, 261–265 (1949)], discusses the nature of the disturbances described by it, and shows that both amplified waves and damped waves are possible. *C. Truesdell* (Bloomington, Ind.).

van Mieghem, Jacques. Le principe d'extremum et la stabilité de certains états de mouvement de l'air atmosphérique. Arch. Meteorol. Geophys. Bioklimatol. Ser. A. 1, 347–357 (1949).

The first theorem proved is that a particle of air which undergoes adiabatic changes of state occupies its position of equilibrium in a mass of air in hydrostatic equilibrium if the sum S of its specific enthalpy and of its potential energy per unit mass has a stationary value. If there is a minimum, the state of hydrostatic equilibrium is stable, if there is a maximum, it is unstable. This theorem is generalised to the case of geostrophic motion and it is shown that the result is still true if there is added to S the longitudinal kinetic energy per unit mass. By the latter term is meant the kinetic energy arising from the component of velocity parallel to the geostrophic flow. The application of the theorem to a permanent circular vortex is worked out.

G. C. McVittie (London).

Queney, P. Adiabatic perturbation equations for a zonal atmospheric current. Tellus 2, 35–51 (1950).

The differential equations governing the infinitesimal adiabatic perturbations of the most general continuous zonal air-flow are set up. The coordinate system employed is a rectangular system of which one axis points to the east, another points northwards tangentially to an isentropic surface and the third is normal to the first two. The dependent variables are the displacements, parallel to these axes, of unit mass of air from the position it would have in the unperturbed motion, and the local disturbance of the pressure. The notation employed is one of extreme complexity. The perturbation equations lead to a differential equation for the pressure disturbance in the case of wave-motion which can be simplified by suitable approximations. Short-period, middle-period and long-period waves are studied in detail. The author concludes by showing that a number of previously published results in this subject are incorrect because of the premature introduction of approximations in the course of deriving the fundamental equations.

G. C. McVittie (London).

Pekeris, C. L. Effect of quadratic terms in differential equations of atmospheric oscillations. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 2314, 10 pp. (1951).

The method employed is to substitute the exponential part of the solution of the linearized equation of motion into the terms quadratic in the velocities, which are neglected in obtaining the solution. A magnitude for the quadratic terms is thus obtained for an isothermal atmosphere and for an atmosphere whose temperature distribution is closely analogous to that observed. It follows that the quadratic terms, in these two cases, equal the linear terms ordinarily

retained, at 130 km. and may even exceed them at greater heights. The qualitative effects on current theory of this conclusion are discussed and it appears that the resonance theory of tidal oscillations cannot be relied upon at the height of the E layer, nor can conclusions drawn from this theory for the levels above the E layer be regarded as certain until the nonlinear tidal equations have been solved.

G. C. McVittie (London).

Woronetz, C. L'effet de l'échauffement sur l'équilibre d'une masse fluide. Acad. Serbe Sci. Publ. Inst. Math. 3, 169–190 (1950).

The author studies a mathematical model for the convection cells over the earth, including the influence of the curvature of the earth (but omitting the influence of the Coriolis force, which might be appreciable if the scale of the motion is of the order of the radius of the earth). Linearized equations are used in line with stability considerations. A criterion is given for the limit of instability when the effect of viscosity is considered. *C. C. Lin.*

Kronig, R., and Thellung, A. On the theory of the propagation of sound in He II. Physica 16, 678–690 (1950).

The phenomenological theory of wave propagations in He II has been considered on the basis of the two-fluid model by including higher order effects heretofore neglected. The method of velocity potentials is used to derive the complex propagation vectors. In addition to the well-known first and second sound, two more wave types are obtained, both strongly damped and arising only at solid surfaces in order to satisfy certain boundary conditions. *L. Tisza.*

de Hoffmann, F., and Teller, E. Magneto-hydrodynamic shocks. Physical Rev. (2) 80, 692–703 (1950).

This paper is devoted to the consideration of plane shock-wave phenomenon in magneto-hydrodynamics. On the assumption that the conductivity of the material is infinite, the fact that there can be no relative motion of the material and the magnetic lines of force leads to the result that the electric field vanishes in the co-ordinate system which is at rest with respect to the fluid. The authors first point out that in magneto-hydrodynamics one can have transverse shocks (i.e., a velocity of propagation perpendicular to the surface of discontinuity) in addition to the usual kind of longitudinal shocks. Two special cases are first considered in detail: the case of parallel shocks when the magnetic field has components only along the direction of propagation (say the x-direction) and the case of perpendicular shocks when the only nonvanishing component of the magnetic field is in a direction (say the y-direction) at right angles to the direction of propagation. In order to discuss the problem in a relativistically invariant way, the authors introduce two co-ordinate systems: a co-moving system (distinguished by a prime) in which the observer is at rest with respect to the medium on the one or the other side of the shock front and the shock system in which the observer is at rest with respect to the shock front and the velocity of flow is v_1 directed toward the shock front on one side and v_2 directed away from the shock front on the other side.

In the case of parallel shocks, the hydrodynamic flow is not coupled with the magnetic lines of force and the shock equations reduce to the relativistic form of the Rankine-Hugoniot equations as given by A. H. Taub [Physical Rev. (2) 74, 328–334 (1948); these Rev. 10, 72]. In the case of perpendicular shocks there is a coupling between the hydrodynamic and the magnetic fields and the shock equations are derived from a consideration of the energy-momentum tensor: In the co-moving system $\mathbf{E}'=0$ and $\mathbf{H}'=(0, H', 0)$. By applying a Lorentz transformation, we find that in the shock system $\mathbf{E}=(0, 0, \beta vH'/c)$ and $\mathbf{H}=(0, \beta H', 0)$ where $\beta=(1-v^2/c^2)^{-\frac{1}{2}}$. Combining the energy-momentum tensor corresponding to these values of $\mathbf{E}$ and $\mathbf{H}$ with that for the hydrodynamical flow, we find that the nonvanishing components of the complete energy-momentum tensor are

$$T_{xx}=\beta^2[\eta'v^2+p'+(1+v^2/c^2)H'^2/8\pi],$$
$$T_{yy}=p'-H'^2/8\pi, \quad T_{zz}=p'+H'^2/8\pi,$$
$$T_{tt}=\beta^2[\eta'+v^2p'/c^4+(1+v^2/c^2)H'^2/8\pi c^2],$$

and

$$T_{xt}=T_{tx}=-\beta^2[v\eta'+vp'/c^2+vH'^2/4\pi c^2],$$

where c denotes the velocity of light, $\eta'c^2=n'(m_0'c^2+E')$ is the relativistic energy density, n' the energy density of particles, m_0' the average rest mass, and E' the excess energy per particle over and above the rest energy. At the shock front the four-dimensional divergence of T must vanish. Since the flow depends only on the x-coordinate, we know that $\partial/\partial y$ and $\partial/\partial z$ are zero and since the shock system is a stationary one, $\partial/\partial t$ is also zero. Hence div $T=0$ requires only that $\partial T_{xx}/\partial x=\partial T_{xt}/\partial x=0$. The application of these conditions at the shock front leads to the result that the components T_{xx} and T_{xt} on the two sides of the shock front must be the same. By writing $\eta=\beta^2\eta'$, $p=\beta^2p'$ and $H=\beta^2H'$ and distinguishing the two sides of the shock front by subscripts 1 and 2, we obtain

$$(1)\quad \eta_1v_1^2+p_1+(1+v_1^2/c^2)H_1^2/8\pi = \eta_2v_2^2+p_2+(1+v_2^2/c^2)H_2^2/8\pi$$

and

$$(2)\quad \eta_1v_1+v_1p_1/c^2+v_1H_1^2/4\pi c^2=\eta_2v_2+v_2p_2/c^2+v_2H_2^2/4\pi c^2.$$

In addition to the foregoing equations we have the equation (3) $\eta_1v_1=\eta_2v_2$ which expresses the conservation of particles. Also since the magnetic lines of force are attached to the particles we must have, in the co-moving system, $H_1'/n_1'=H_2'/n_2'$ or since $n'=n/\beta$, we have (4) $H_1/n_1=H_2/n_2$ or $H_1v_1=H_2v_2$. Equations (1), (2), (3) and (4) provide the required generalization of the Rankine-Hugoniot equations for perpendicular shocks in magneto-hydrodynamics. The authors discuss these equations in various limiting forms and establish among others the following results: (i) In the extreme relativistic case the equation giving the shock velocity in the limit of vanishing discontinuity is $(3v^2-c^2)(v^2-c^2)=9H'^2/16\pi\eta'c^2$ and, in particular for $H'=0$, $v=c/\sqrt{3}$. (ii) For all ordinary forms of the equation of state $v\to c$ only when the energy density of the magnetic field becomes infinitely large compared to the material energy density. (iii) In the nonrelativistic case, the velocity of sound is given by $v=[(\partial p/\partial v)/(\partial\rho_0/\partial v)+H^2/4\pi\rho_0]^{\frac{1}{2}}$ and, quite generally, in this case the shock equations are exactly of the same form as classical Rankine-Hugoniot equations provided we replace p and E in these equations by $p^*=p+H^2/8\pi$ and $E^*=E+H^2/8\pi\rho$.

The paper concludes with a discussion of oblique shocks. If the component T_{xy} of the energy-momentum tensor is not zero then in the limit of vanishing discontinuity, it is shown that the square of the velocity of propagation satisfies a quadratic equation and that if $(v_x)_+^2$ and $(v_y)_-^2$ are the two solutions then $(v_x)_+^2+(v_x^2)_-=(dp/d\rho)+H^2/4\pi\rho$ and $(v_x)_+(v_x)_-=(H_x^2/4\pi\rho)(dp/d\rho)$, i.e., the sum of the squares and the product of the ordinary sound velocity and the velocity of a magneto-hydrodynamic wave. The case when

$T_{xy}=0$ is more general and leads to a new class of solutions the properties of which are briefly discussed.

S. Chandrasekhar (Williams Bay, Wis.).

Elasticity, Plasticity

Aquaro, Giovanni. Un teorema di media per le equazioni dell'elasticità. Rivista Mat. Univ. Parma **1**, 419–424 (1950).

The author proves the following theorem of mean value. Let s_i be a vector satisfying Navier's statical equation throughout a sphere whose surface is Σ, whose center is at P_0, and whose radius is R, extraneous forces being zero. Let P be an interior point with coordinates x_i, let Q be a boundary point, and let the quantities G_{ij} be defined by

$$G_{ij}=\delta_{ij}+\frac{R^2(k-1)}{2(k+2)}\left[\frac{\partial^2}{\partial x_i\partial x_j}\frac{P_0Q^2-P_0P^2}{PQ^2}\right]_{P=P_0},$$

while the definition of k in terms of the Lamé constants is $k=2+\lambda/\mu$. Then the value of s_i at P_0 is

$$s_i=\frac{1}{4\pi R^2}\oint_\Sigma G_{ij}s_j\,dS.$$

C. Truesdell (Bloomington, Ind.).

Lodge, A. S. The compatibility conditions for large strains. Quart. J. Mech. Appl. Math. **4**, 85–93 (1951).

This paper is the fifteenth since 1902 to derive the compatibility equations for large strain. In the review of the fourteenth [Seugling, Amer. Math. Monthly **57**, 679–681 (1950); these Rev. **12**, 555] references to the first twelve are given, the thirteenth (cited by the author) having been omitted by oversight. *C. Truesdell* (Bloomington, Ind.).

Castoldi, Luigi. Linee sostanziali inestese nelle deformazioni finite dei continui materiali. Atti Accad. Ligure **6**, 165–169 (1950).

The author sets up differential equations for curves suffering no change in length in a finite deformation of a continuum. Since these are of Monge's type, by using known theorems he is able to make a few statements regarding the unextended curves. *C. Truesdell* (Bloomington, Ind.).

Rivlin, R. S., and Thomas, A. G. Large elastic deformations of isotropic materials. VIII. Strain distribution around a hole in a sheet. Philos. Trans. Roy. Soc. London. Ser. A. **243**, 289–298 (1951).

In continuation of Rivlin's program of obtaining significant exact solutions of the general equations of elasticity for strains of any magnitude and any form of strain energy in an incompressible body, the authors consider an annular slab subject to uniform radial tension on its outer surface. In this case, the solution is not exhibited, but a simple iterative procedure for calculation is presented. Very good agreement with experimental results over a wide range of deformations is obtained. *C. Truesdell.*

Hassé, H. R. The bending of a uniformly loaded clamped plate in the form of a circular sector. Quart. J. Mech. Appl. Math. **3**, 271–278 (1950).

The paper deals with the approximate determination of the deflection of a uniformly loaded plate in the form of a circular sector whose boundary is clamped. In other words, with the approximate determination of the function $w(x, y)$, where $\Delta\Delta w=q/D$ in the interior S (here, $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$ and q and D are given constants) and $w=\partial w/\partial n=0$ on the boundary C of the domain. The method employed is that of A. Weinstein, originally developed for the purpose of calculating lower bounds for the eigenvalues of clamped plates [Étude des spectres des équations aux dérivées partielles . . . , Mémor. Sci. Math., no. 88, Gauthier-Villars, Paris, 1937]; later applied to the statical case of the deflection of a clamped plate [A. Weinstein and D. H. Rock, Quart. Appl. Math. **2**, 262–266 (1944); these Rev. **6**, 67]; the latter application having been theoretically justified by Weinstein and Jenkins [Trans. Roy. Soc. Canada. Sec. III. (3) **40**, 59–67 (1946); these Rev. **8**, 547]. Briefly, the method is as follows. Let m be a nonnegative integer, and $v_0, v_1, \cdots, v_m$ be $m+1$ biharmonic functions which vanish on the boundary (i.e., $\Delta\Delta v_n=0$ in S and $v_n=0$ on C for $n=0, 1, \cdots, m$); and let W satisfy $\Delta\Delta W=0$ in S, and $W=\Delta W=0$ on C. As the "mth approximation" to the deflection w one takes the function $W+\sum_{n=0}^{m}A_nv_n$, where the real numbers $A_0, A_1, \cdots, A_m$ are the solution of the system of $m+1$ linear equations

$$\sum_{s=0}^{m}A_s\cdot\int_S\Delta v_s\cdot\Delta v_n\,dxdy=-\int_S v_n\cdot[q/D]dxdy=-\int_S\Delta v_n\cdot\Delta W dxdy,$$

$n=0, 1, \cdots, m$. The numerical calculations are carried out explicitly for the special case of a semicircular sector of unit radius, $|\theta|\leqq\frac{1}{2}\pi$, $r\leqq 1$, taking $q/D=1$, and $m=0, 1, \cdots, 6$.

J. B. Diaz (College Park, Md.).

Zerna, W. Berechnung an den Rändern belasteter, allgemeiner Schalen. Z. Angew. Math. Mech. **30**, 370–374 (1950). (German. English, French, and Russian summaries)

This paper is an extremely condensed presentation of a new theory of the bending of shells. The objective is to obtain sufficiently accurate equations easier to solve than those of Love. Instead of the stress resultants, stress couples, and middle surface displacements which are the variables of Love's theory, the author introduces somewhat more elaborate weighted averages of the stresses and displacements. The analysis is presented in tensor notation in general coordinates. In place of the assumptions of Love's theory the author states that he supposes only that the thickness of the shell is very small compared to the minimum of the radii of curvature and the characteristic dimensions of the shell. Since all calculations are omitted, one cannot always see precisely how this assumption is used. The theory is to be explained in full in a forthcoming work by A. E. Green and the author. In § 3 the author applies his equations to the case of a shell loaded only at its edges. By introducing several further approximations he reduces the problem to solution of a complex fourth order partial differential equation. *C. Truesdell* (Bloomington, Ind.).

Vedeler, Georg. Basic function for beams with arbitrary constraint. Norske Vid. Selsk. Forh., Trondheim **22**, no. 37, 171–177 (1950).

The deflection of a beam with elastically constrained ends is expressed in terms of the eigenfunctions associated with the beam oscillations. A table and a graph illustrate the eigenfunctions associated with various degrees of constraint.

G. F. Carrier (Providence, R. I.).

Fadle, J. Der homogene, durchlaufende Träger auf unverschieblichen Stützen. Ing.-Arch. 17, 317–335 (1949).

Several problems associated with a continuous beam on uniformly spaced supports are considered. Extensive tables represent the results. The problems differ in the choice of end conditions and number of supports. *G. F. Carrier.*

Edwards, R. H. Stress concentrations around spheroidal inclusions and cavities. J. Appl. Mech. 18, 19–30 (1951).

The work of Sadowsky and Sternberg [same J. 14, A-191–A-201 (1947); these Rev. 9, 121] is extended in several ways. Using the same method of three harmonic displacement functions, closed solutions are obtained for an elastic body which is under any uniform state of stress at infinity and which contains a spheroidal cavity or elastic inclusion. Complete bond is assumed between the inclusion and the body. Numerical results and graphs are given for the case of uniaxial tension at infinity in a direction perpendicular to the axis of symmetry of the spheroid. It is interesting that the inclusion has a homogeneous state of stress. The spheroid is varied from essentially a line or cylinder to a sphere and the inclusion from a rigid inclusion to a free cavity. Poisson's ratio is taken as 0.3. The thermal stress problem is also solved for the inclusion at one constant temperature and the surrounding body at another.

D. C. Drucker (Providence, R. I.).

Ayre, R. S., Ford, George, and Jacobsen, L. S. Transverse vibration of a two-span beam under action of a moving constant force. J. Appl. Mech. 17, 1–12 (1950).

A simply supported two-span beam is acted upon by a force which moves along the beam at constant speed. The analysis of this problem leads to a series type solution. This solution is interpreted for various force traversing speeds and a resonance phenomenon is noted. An experiment with a mechanical model is reported and good agreement with the theory is obtained. *G. F. Carrier.*

Mindlin, Raymond D. Thickness-shear and flexural vibrations of crystal plates. J. Appl. Physics 22, 316–323 (1951).

Commencing with the three-dimensional stress-displacement and stress-acceleration equations for a body whose elastic properties have monoclinic symmetry, the author develops a theory for the vibration of crystal plates as a special case. The frequencies of thickness-shear vibrations are determined for an infinite crystal plate; these agree with the results of I. Koga [Physics 3, 70–80 (1932)]. In order to discuss the vibrations of rectangular plates of finite length, additional modes are combined with those for the infinite plate; this results in the excitation of flexural modes. The boundary conditions for finite width are not satisfied, but there is experimental evidence to show that the effect of finite width on the resonant frequencies is unimportant. The transcendental frequency equation derived is then compared with the experimental results of R. A. Sykes [R. A. Heising, Quartz Crystals for Electrical Circuits . . . , Van Nostrand, New York, 1946, Chapter 6]. Very good agreement is reached. A method of solving the frequency equation by successive approximations is discussed, this being useful for obtaining the frequencies of higher modes. Algebraic formulas for the zeros and infinities of the frequency equation are given, as is also the method for obtaining the mode shapes. *H. D. Conway* (Ithaca, N. Y.).

Mettler, E. Allgemeine Theorie der Stabilität erzwungener Schwingungen elastischer Körper. Ing.-Arch. 17, 418–449 (1949).

This paper presents an integration theory for the stability problem previously formulated by the author [Ing.-Arch. 16, 135–146 (1947); these Rev. 10, 87]. The problem is concerned with the stability of the transients which arise in elastic systems under forces which are periodic in the time. The direct method of the variational calculus is applied to this variational problem and it degenerates into a family of simultaneous Hill type equations. A detailed consideration of the stability criteria for such systems is presented and some interesting specific examples are treated.

G. F. Carrier (Providence, R. I.).

***March, H. W. Elastic stability of the facings of sandwich columns.** Proc. Symposia Appl. Math. v. 3, pp. 85–106. McGraw-Hill Book Co., New York, N. Y., 1950. $6.00.

The stability of two elastically supported sheets which undergo sinusoidal wave patterns is considered. The wavelength is so chosen that the compressive wrinkling stress is a minimum for the type of instability considered. The core material is assumed to be an orthotropic elastic medium either in a state of plane stress or plane strain. The following assumptions, similar to those of Gough, Elam, and de Bruyne [J. Roy. Aeronaut. Soc. 44, 12–43 (1940)] are made: (1) the tangential component of displacement at the glue line is taken to be zero; (2) the core has been considered attached directly to the middle surface of the facings in determining the equilibrium equations; (3) the distabilizing effect of compressive stress in the core has been neglected.

The critical stresses for antisymmetrical and symmetrical wrinkling are determined. For the former case a "terminal point" is found below which this type of instability cannot occur. Such is not the case for symmetrical wrinkling. Numerical computations show that the author's results are more conservative than those of Williams, Leggett, and Hopkins [Rep. no. 2, Structures, Roy. Aircraft Est., 1947; Report no. A. D. 3174, Roy. Aircraft Est., 1941], which do not make the first two assumptions. The effect of compressive stress in the core is expected to be small. Stresses in the core when the faces are not initially flat are also considered. *G. H. Handelman* (Pittsburgh, Pa.).

Kondo, Kazuo. On the fundamental equations of the theory of yielding. J. Jap. Soc. Appl. Mech. 3, 184–188 (1950).

The author states that there is a close analogy between buckling and yield. He regards yield as a small displacement w of a three-dimensional region "in a direction normal to the original flat position," as seen from the six-dimensional Euclidean space which envelops the three-dimensional Riemannian (but non-Euclidean) space which the author states the yielding material occupies. He obtains the equation

$$B\nabla^4 w+\sigma^{ij}\frac{\partial^2 w}{\partial x^i \partial x^j}=0,$$

where σ^{ij} is the stress tensor in the yielding material and B is a constant chosen so as to make the work done a positive quantity. Boundary conditions are given also. The reviewer is unable to follow the reasoning leading to these results or to see how a solution w is to be interpreted or used.

C. Truesdell (Bloomington, Ind.).

Kondo, Kazuo. The mathematical analysis of the yield point. I. Uniform stress. J. Jap. Soc. Appl. Mech. **3**, 188–195 (1950).

The theory developed in the paper reviewed above is applied to a special case. The author states that "w is to go rapidly to a state of uniformity as we go into the inner part far from the surface" and that "the incompatible deformation must be of a periodic distribution outside the boundary layer." He considers a solution of exponential type, discusses the wavelength and the numerical values of certain quantities, and claims to show that his results are "coincident with the theory of maximum shear at least for the uniform stress," whence he concludes that "correctness of our theory may be considered well proved." The reviewer is unable to follow the analysis or to understand what the results are intended to show. *C. Truesdell* (Bloomington, Ind.).

Hayasi, Hirosi. On the resistance experienced by a body which moves along the plane surface of an elastic solid. J. Phys. Soc. Japan **2**, 196–198 (1947).

In previous papers [Proc. Phys.-Math. Soc. Japan (3) **24**, 533–548, 800–808, 915–922 (1942); **25**, 47–56, 198–206, 391–395, 468–480, 648–658 (1943); these Rev. **7**, 503] the author has computed the complex deformation D_x, D_y, D_z of the boundary $z=0$ of the solid $z\leqq 0$ in 3 space, when a disturbance is propagated with uniform velocity and the induced pressure P is uniformly distributed over a circular area. The writer evaluates $\int\int_S P(\partial/\partial t)\Re D_z dxdy$ where S is $x^2+y^2\leqq a^2$ and $\Re$ indicates the real part. This is interpreted as proportional to the resistance occasioned by energy dissipation. *D. G. Bourgin* (Urbana, Ill.).

Fu, C. Y. Free vibrations of an inner stratum. Acad. Sinica Science Record **3**, 81–85 (1950). (English. Chinese summary)

Normal modes of vibration of a plane thin stratum imbedded in an elastic (infinite) medium are examined and the conditions under which they occur are studied. The problem discussed is important for seismic prospection in regions where thin strata are found imbedded in thick formations (example: shaly areas). *E. Kogbetliantz.*

Caloi, Piétro. Comportement des ondes de Rayleigh dans un milieu firmo-élastique indéfini. Publ. Bureau Central Seismol. Internat. Sér. A. Trav. Sci. **17**, 89–108 (1950).

L'auteur étudie les ondes de Rayleigh à la surface d'un milieu où le frottement intérieur est du type de Voigt, autrement dit un milieu firmo-visqueux au sens de Jeffreys. De nombreux auteurs outre ceux que cite l'auteur, notamment Arakawa [Geophys. Mag. **4**, 215–224 (1931)], Sezawa et Kanai [Bull. Earthquake Res. Inst. Tokyo **16**, 491–503 (1938)], avaient déjà montré que les équations correspondantes, soit:

$$\rho\partial^2 U/\partial t^2=[(\lambda+\mu)+(\lambda'+\mu')\partial/\partial t]\,\mathrm{grad}\,\theta+(\mu+\mu'\partial/\partial t)\nabla^2 U,$$

où U est le vecteur déplacement, θ la dilatation, conduisaient pour les ondes de Rayleigh à une dispersion anomale qui semble peu compatible avec les observations. L'auteur prend le problème au début. On trouvera dans son article des calculs numériques détaillés pour le cas où $\lambda=\mu$ et $3\lambda'+2\mu'=0$. *J. Coulomb* (Paris).

Swida, W. Die Plattengleichungen für den elastisch-plastischen Zustand. Z. Angew. Math. Mech. **30**, 375–381 (1950). (German. English, French, and Russian summaries)

A differential equation is derived for the deflection of an elastic plastic plate under the assumptions: (a) straight lines normal to the middle surface before bending remain straight and normal, (b) the material is ideally plastic and obeys the Mises yield condition, (c) all components of stress are continuous so that the elastic stresses determine the plastic. Two very special cases are solved, cylindrical bending and homogeneous bending. *D. C. Drucker.*

Symonds, P. S. Shakedown in continuous media. J. Appl. Mech. **18**, 85–89 (1951).

A simple proof is given for Melan's theorem that a body under variable loading will "shake down" if there exists any residual state of stress $\bar{\rho}$ in equilibrium such that the addition of elastic stresses produced by the loading will not exceed the yield limit at any point. The Mises yield criterion is assumed. Proof consists of showing that the actual residual state ρ always approaches the state $\bar{\rho}$ in the sense that the elastic strain energy of $\bar{\rho}-\rho$ continually decreases if yielding occurs. The theorem is applied to the case of a circular bar subjected to an axial force and a twisting moment which vary independently between given limits.
D. C. Drucker (Providence, R. I.).

Hill, R., Lee, E. H., and Tupper, S. J. A method of numerical analysis of plastic flow in plane strain and its application to the compression of a ductile material between rough plates. J. Appl. Mech. **18**, 46–52 (1951).

It is assumed that plastic strain is so large that elastic strains can be neglected, that the material exhibits a well-defined yield point, and negligible work hardening. The slip lines $\alpha=$const., $\beta=$const. have radii of curvature R, S related by $dS+Rd\phi=0$ along $\alpha=$const., $dR-Sd\phi=0$ along $\beta=$const., where $\phi=\arctan\partial\alpha/\partial x$. The numerical method used replaces these differential forms by difference relations, and the slip lines are constructed by starting from known conditions at a boundary. Velocities are found in like manner. *R. E. Gaskell* (Seattle, Wash.).

Leibfried, Günther. Über die auf eine Versetzung wirkenden Kräfte. Z. Physik **126**, 781–789 (1949).

Die Theorie der plastischen Deformation beruht auf die durch äussere Kräfte verursachten Bewegungen der Versetzungen, die ja das plastische Fliessen verursachen. Der Verfasser berechnet deshalb die Kräfte welche äussere Spannungen auf solch eine Versetzung ausüben. Eine Versetzung definiert man dabei folgendermassen: Längs der Versetzungslinie $x=\xi$, $y=\eta$ schneidet man den Kristall entlang der x-Richtung auf, verschiebt die zwei Spaltflächen S zueinander um die Gitterkonstante λ und verbindet sie wieder miteinander in dieser neuen Lage. Die Oberfläche O des Kristalls bleibt dabei kräftefrei. Bei der Erzeugung der Versetzung muss man die Arbeit $E^v(\xi,\eta)$ leisten. Bringen wir jetzt an der Oberfläche äussere Kräfte an, so berechnet sich die Arbeit E^a, die dabei zu leisten ist, ganz so, wie wenn die Versetzung garnicht vorhanden wäre, weil ja O kräftefrei ist. Also folgt für die ganze elastische Energie $E=E^v(\xi,\eta)+E^a$.

Bringt man jetzt die zwei Deformationen in umgekehrter Reihenfolge an, so muss man die selbe elastische Energie erhalten und demzufolge verschwindet das Wechselwirkungsglied $\int_{O+S}(\mathfrak{d}^v,\mathfrak{P}^a d\mathfrak{f})$, wo $\mathfrak{d}$ der Verschiebungsvektor und $\mathfrak{P}$ der Spannungstensor ist. Diese zwei Gleichungen sind die Grundlagen der Theorie. Für die Kraftkomponenten auf die Versetzung folgt $K_x=-\partial E/\partial\xi$ und $K_y=-\partial E/\partial\eta$. Weitere Rechnungen liefern $K_x=\lambda\int dz p^a{}_{xy}(\xi,\eta,z)$ und eine analoge Formel für K_y. Hier bedeutet $P^a{}_{xy}$ die (x,y)-Komponente des von den äusseren Kräften verursachten Spannungstensors. Im Falle von inneren Spannungen erhält

man ein analoges Resultat, wenn die von einer anderen Versetzung herrühren, jedoch nicht, wenn sie von einem Fremdatom verursacht werden. *T. Neugebauer.*

Leibfried, Günther, und Dietze, Horst-Dietrich. Zur Theorie der Schraubenversetzung. Z. Physik **126**, 790–808 (1949).

Als Schraubenversetzung definieren die Verfasser eine Deformation des kubisch primitiven Kristalls, bei der eine Atomreihe im Verhältnis zu der darunterliegenden in Richtung dieser Atomreihe [z-Achse] um die Gitterkonstante λ verschoben wird. Diese Störung wandert dann in den Kristall hinein. Die Behandlung dieses Problems ist besonders einfach, weil nur Verschiebungen w entlang der z-Richtung auftreten. Der Spannungstensor P_{ik} hat dann nur zwei Komponenten p_{xz} und p_{zy}. Im statischen Fall genügt w der zwei-dimensionalen Potentialgleichung $\Delta w=0$ und im zeitabhängigen der Gleichung $\Delta w=c^{-2}\partial^2 w/\partial t^2$. Dabei ist $c=(g/\rho)^{\frac{1}{2}}$ die Schallgeschwindigkeit der Scherungswellen, g ist der Schubmodul und ρ die Dichte. Die Elastizitätstheorie liefert $p_{xz}=g\partial w/\partial x$ und $p_{zy}=g\partial w/\partial y$. Näherungsweise kann man schreiben $P_{xz}=g(w_A-w_B)/\lambda$, wenn die Verschiebungen der benachbarten Atomreihen gering sind. Im entgegengesetzten Fall benützte Peierls den Ansatz $p_{xz}=(g/2\pi)\sin 2\pi(w_A-w_B)/\lambda$. Durch weitere Umformungen kann man diese Gleichung in die Form

$$\tau_A=-(g/2\pi)\sin 4\pi w_A(x)/\lambda$$

bringen, wo jetzt τ_A die Spannungskomponente p_{xz} in diesem Falle bedeutet. Diese Grösse erhält man jedoch auch, wenn man die Potentialgleichung für w löst und daraus p_{xz}, bzw. τ_A, berechnet. Gleichsetzen dieser zwei Resultate liefert den Zusammenhang $\sin 4\pi w_A(x)/\lambda=2\int_{-\infty}^{+\infty}(x-\xi)^{-1}w_A'(\xi)d\xi$, den die Verfasser Peierlssche Integralgleichung nennen. Sie besitzt die Lösung $w_A=(\lambda/2\pi)\arctan 2x/\lambda$ für eine Versetzung. Alle diese Resultate beziehen sich auf einen unendlich ausgedehnten Kristall, für eine Platte wird die Peierlssche Gleichung und deren Lösung etwas verwickelter. Nach der einfachen elastischen Methode erhält man die Lösung für eine Platte aus der für den unendlichen Kristall durch wiederholte Spiegelungen an den Grenzflächen als eine unendliche Reihe.

Eine bewegte Versetzung kann man ganz analog behandeln, wenn man $x-vt=\xi$ setzt. Statt $\Delta w=0$ hat man dann $\beta^2\partial^2 w/\partial\xi^2+\partial^2 w/\partial y^2=0$, wo $\beta=(1-v^2/c^2)^{\frac{1}{2}}$ ist. Aus den Rechnungen folgt, dass Versetzungen praktisch immer mit Schallgeschwindigkeit laufen müssten. Daraus folgt die Notwendigkeit der Annahme einer Dämpfung. Im Anhang werden einige Integrale berechnet. *T. Neugebauer.*

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Signorini, Antonio. Qualche teorema di ottica geometrica. Rend. Sem. Mat. Fis. Milano **20** (1949), 1–12 (1950).

The author investigates the classical condition that in an isotropic medium all points have a sharp image and derives the theorem of Bouguer for concentric layers [cf. the paper by the reviewer, Z. Instrumentenkunde **53**, 436–443 (1933)]. *M. Herzberger* (Rochester, N. Y.).

Staeble, F. Abbildung von Dingpunkten, die im Hauptschnitt einer Folge liegen. Optik **8**, 1–23 (1951).

The author gives an analytical discussion of the aberrations of a wave front for an optical system with plane symmetry, where the object point lies in the symmetry plane. Such a theory covers the image of an axis point in torical systems as well as the image of an off axis point in systems with rotation symmetry. The reader should be reminded that the author's conclusions are based on an approximation of the wave surface up to terms of fifth order; the geometrical statements of the author are therefore approximations, whereas the analytical representation will in most cases be sufficient. *M. Herzberger* (Rochester, N. Y.).

Köhler, H., und Pradel, G. Eine neue Interpolationsmethode zur Ermittlung der gesamten Zerstreuungsfigur zentrierter optischer Systeme mit unendlich fernem Bild. Z. Angew. Math. Mech. **31**, 47–53 (1951). (German. English, French, and Russian summaries)

The authors develop formulas to calculate the light distribution in the image in an optical system. To avoid tracing skew rays they compute the first order skew aberrations by tracing sagittal astigmatism along the meridional rays. They thus obtain the equivalent of the fifth order aberrations of an optical system, determining the mixed characteristic function V to the corresponding accuracy. [The reviewer suggests, as a practical improvement of the author's method, tracing more rays than necessary and determining the coefficients of the characteristic function by least square methods.] *M. Herzberger* (Rochester, N. Y.).

Brdička, Miroslav. The reflexion of light from glass with a transparent homogeneous surface-layer. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **48** (1947), 63–70 (1950).

The problem indicated by the title can be reduced to the solution of a single complex equation, which, however, cannot be solved by algebraic methods. The present paper transforms this equation to a real equation of fifth degree, whose real roots can then be approximated to any desired accuracy. *M. Herzberger* (Rochester, N. Y.).

Brdička, Miroslav. The reflexion of light from glass with a natural transparent inhomogeneous surface-layer. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **49** (1948), 81–89 (1950).

In the study of the reflection of plane polarized light many reflecting substances exhibit a feeble elliptical polarization which is explained by assuming that a thin transparent surface-layer is spontaneously formed. Drude [Lehrbuch der Optik, Hirzel, Leipzig, 1900, p. 275] has suggested a formula for the polarization if the dielectric constant of the layer is given as a function of its distance from the glass surface. The author suggests a special formula for this dependency and then computes from Drude's formula the polarization. The results are compared with experimental data obtained by the author. *M. Herzberger.*

Wolf, E. Light distribution near focus in an error-free diffraction image. Proc. Roy. Soc. London. Ser. A. **204**, 533–548 (1951).

Si abbia un'onda sferica di raggio f, proveniente da un'apertura circolare di raggio R. L'autore si propone di valutare la frazione della luce totale che cade al di dentro di

un cerchio di raggio r centrato sull'asse, in un piano non coincidente con quello focale. Partendo dalle espressioni di Lommel egli ottiene la soluzione sotto forma di serie rapidamente convergente. Nel caso che r coincida con il raggio di confusione geometrico, la frazione cercata assume la forma semplice: $1-J_0(\nu)\cos\nu-J_1(\nu)\sin\nu$, essendo $\nu=2\pi Rr/\lambda f$. *G. Toraldo di Francia* (Firenze).

Cimmerman, G. K. Refraction in observations through a solar screen. Akad. Nauk. SSSR. Astr. Žurnal **27**, 257–266 (1950). (Russian)

En supposant (1) un champ des températures stationnaire dans un champ gravifique uniforme, (2) une vitesse de convection turbulente proportionnelle au gradient de la température, l'auteur considère deux types d'atmosphères: l'une limitée à la base inférieure par un plan horizontal isotherme, "atmosphère normale"; et une autre limitée par une surface cylindrique à génératrices horizontales, "atmosphère déformée." L'auteur montre qu'on peut établir entre ces deux surfaces une correspondance biunivoque; on établit que les températures correspondantes, T et T', satisfont à une même équation aux dérivées partielles et sont égales. Il en est de même, pratiquement, pour les pressions. En posant, en outre, qu'aux points correspondants (x, y, z) et (x', y', z') les deux indices de réfraction, μ et μ', sont pratiquement égaux, on trouve que les points de A' qui correspondent aux points de A forment aussi une surface optique. On déduit ensuite une relation entre les distances zénithales observées dans les deux atmosphères. Étant donné que dans les pavillons méridiens actuels les écrans solaires ont la forme d'une surface cylindrique à axe horizontal, on peut considérer l'atmosphère au-dessous de l'écran comme une "atmosphère déformée" et appliquer la formule de l'auteur. Les résultats d'observation à l'Observatoire de Nikolaev, pendant les années 1929–1939 ont bien confirmé la validité de la formule. *M. Kiveliovitch* (Paris).

Gans, Ricardo, and Beck, Guido. Diffraction of light at a sharp edge. Revista Unión Mat. Argentina **14**, 425–443 (1950). (Spanish. German summary)

L'autore parte dalla soluzione di Sommerfeld per il semipiano conduttore e con un metodo di perturbazione tratta il caso in cui il semipiano si trasforma in una lamina con orlo arrotondato, essendo il raggio di curvatura molto piccolo. *G. Toraldo di Francia* (Firenze).

Jones, D. S. Note on diffraction by an edge. Quart. J. Mech. Appl. Math. **3**, 420–434 (1950).

Assuming that the electric current and charge on a perfectly conducting sheet S are L-integrable in the neighborhood of the edge C, the author shows that (1) no excess electric line current can exist on C, and (2) the component of the surface current density normal to C vanishes on C. These two properties follow from the boundary conditions $H_{\text{norm}}=E_{\text{tan}}=0$ on S. He further postulates the absence of excess line distributions of magnetic current and charge on C, and derives $E_s=o(1)$, $H_s=\text{const.}+o(1)$, all other components $o(1/\rho)$, where s and ρ measure the distances along and perpendicular to C respectively. This enables the author to derive a uniqueness theorem for the diffraction problem under consideration [see also J. Meixner, Ann. Physik (6) **6**, 2–9 (1949); these Rev. **11**, 562]. Further, assuming a special form for the surface current density (which satisfies the previous requirements) the author shows that, so far as the lowest possible terms are concerned, $I_\rho\propto\rho^{\frac{1}{2}}$, $I_s\propto\rho^{-\frac{1}{2}}$ [see also A. W. Maue, Z. Physik **126**, 601–618 (1949); these Rev. **11**, 293]. Finally the diffraction, in three dimensions, of a plane wave by a semi-infinite plane is obtained [cf. E. T. Copson, Proc. Roy. Soc. London. Ser. A. **202**, 277–284 (1950); these Rev. **12**, 223]. *C. J. Bouwkamp.*

Bouwkamp, C. J. On Bethe's theory of diffraction by small holes. Philips Research Rep. **5**, 321–332 (1950).

Let plane polarised electromagnetic waves travelling in the positive z direction be incident on a perfectly reflecting screen in the plane $z=0$ and be diffracted through a small circular hole whose radius a is small compared with the wave-length. The total field $\mathbf{E}$, $\mathbf{H}$ must then satisfy the following conditions: (i) the tangential electric and normal magnetic forces vanish on the screen; (ii) the total field is continuous through the hole. If the incident field is $\mathbf{E}_0=\{2i\sin kz, 0, 0\}$, $\mathbf{H}_0=\{0, 2\cos kx, 0\}$, a time factor $e^{-i\omega t}$ being understood, these conditions reduce to $E_x=E_y=H_z=0$ on the screen, and $H_x=E_z=0$, $H_y=1$ in the hole. This problem was solved approximately by Bethe [Physical Rev. (2) **66**, 163–182 (1944); these Rev. **6**, 165] by introducing a fictitious magnetic charge and current sheet in the hole. It is shown here that Bethe's solution gives $E_z=ikx$ in the hole, which is not zero to the order of approximation considered. Thus Bethe's solution is wrong. The author also gives another argument which demonstrates the inaccuracy of Bethe's solution. Meixner [Ann. Physik (6) **6**, 2–9 (1949); these Rev. **11**, 562] showed that the total field must have a singularity on the rim of the hole: The electric force becomes infinite there and is normal to the rim. Bethe's solution has no such singularity. The error in Bethe's approximation is corrected in the present paper. Although the error does not invalidate the formulae for the distant field, the field near the hole differs appreciably from that predicted by Bethe.

Reviewer's note: The reviewer [Proc. Roy. Soc. London. Ser. A. **186**, 100–118 (1946); these Rev. **8**, 179] obtained Bethe's approximation from a formulation of plane diffraction problems by means of differential-integral equations; and the author's criticism applies equally to this work. The author, in his review of the cited paper, criticised the differential-integral equation method as being valid only when the component of the total electric force tangential to the rim of the hole vanished, a condition with Meixner has since shown must be satisfied. In a subsequent paper [Proc. Roy. Soc. London. Ser. A. **202**, 277–284 (1950); these Rev. **12**, 223] the reviewer introduced a simpler system of differential-integral equations (essentially by eliminating the scalar potentials) to which the author's criticism would not apply. But at the same time, the reviewer claimed that the criticism itself was unsound in that it applied Green's transformation to singular integrals. The author has since convinced the reviewer that this was in error; the reviewer had overlooked the fact that it is possible for the electric vector to be infinite on the rim and yet have a vanishing tangential component. *E. T. Copson* (St. Andrews).

Bouwkamp, C. J. On the diffraction of electromagnetic waves by small circular disks and holes. Philips Research Rep. **5**, 401–422 (1950).

This paper deals with the diffraction of a plane polarised electromagnetic wave by a conducting circular disc for the case of normal incidence. This problem has recently been solved in a very satisfactory way by Meixner and Andrejewski [Ann. Physik (6) **7**, 157–168 (1950); these Rev. **12**, 145] who used oblate spheroidal wave functions. The author solves the same problem by means of the differential-integral equations for the currents in the disc. These equations are

solved approximately on the assumption that ka is small, where a is the radius of the disc, k the wave number. Six terms of a power-series solution in the basic variable ka are derived. The scattered field on the surface of the disc and the field in the wave zone are calculated to the same degree of accuracy. The corresponding solution for the diffraction of a plane wave by a circular hole is obtained by an application of Babinet's principle. *E. T. Copson.*

Booker, H. G., Ratcliffe, J. A., and Shinn, D. H. Diffraction from an irregular screen with applications to ionospheric problems. Philos. Trans. Roy. Soc. London. Ser. A. **242**, 579–609 (1950).

In this paper an analysis is made of the diffraction effects produced when a plane wave is incident upon an irregular diffracting screen with a view to applications to the problem of the reflexion of radio waves from an ionosphere which is irregular in the horizontal plane. The nature of the irregular screen is assumed to be given in terms of the electric wave field $E(x, 0)$ in a plane just beyond the screen and that variations occur over the plane in the x direction only. If $E(x, 0)$ is represented as a Fourier integral in the form $E(x, 0) = \int_{-\infty}^{+\infty} P(s)\{\exp(2\pi isx/\lambda)\}ds/\lambda$ (where λ denotes the wave-length), the author defines $P(s)$ as the angular spectrum of the wave disturbance. The wave disturbance at a point z away from the plane is then given by

$$E(x, z) = \int_{-\infty}^{+\infty} P(s) \exp\{2\pi iz\sqrt{(1-s^2)}\} \exp(2\pi isx)ds/\lambda.$$

Since $E(x, 0)$ and $P(s)$ are related as Fourier transforms it follows that the auto-correlation coefficient,

$$\rho_E(\xi) = \int_{-\infty}^{+\infty} E^*(x, 0)E(x+\xi, 0)dx \Big/ \int_{-\infty}^{+\infty} |E(x, 0)|^2 dx,$$

is related to the power spectrum $|P(s)|^2$ by

$$\rho_E(\xi) = \int_{-\infty}^{+\infty} |P(s)|^2 \exp(2\pi is\xi)ds \Big/ \int_{-\infty}^{+\infty} |P(s)|^2 ds.$$

This relation is generalized to the case when a large number of sample irregular screens each limited to an optical "stop" of the same size are examined. It is then shown that (i) the angular power spectrum $|P(s)|^2$ averaged over systems is the Fourier transform of the auto-correlation function $\overline{\rho_E(\xi)}$ of the disturbance at the screen averaged over systems and its width depends on the "fineness of the grains" of the screens; (ii) the auto-correlation function $\overline{\rho_P(\sigma)}$ of the spectrum $P(s)$, averaged over samples, is determined by the width of the aperture, and is the same as the angular spectrum of the aperture itself.

The Fresnel diffraction pattern given by $E(x, z)$ is next considered and it is shown in particular that in the optical case when the term in $\exp\{2\pi iz\sqrt{(1-s^2)}\}$ can be neglected for $s>1$, the auto-correlation function of the Fresnel diffraction pattern is the same as that of the field distribution itself. And when there is a random distribution of field over the screen, it is shown that the auto-correlation function of the amplitude over the Fresnel diffraction pattern formed by a random screen is the same over all such patterns provided that observations are made at distances greater than a few wave-lengths from the screen. The auto-correlation function is also the same as that over the screen itself when either (a) the screen has no structure comparable with a wave-length or (b) structure of this size is neglected in computing the auto-correlation. The problem of the irregular "fading" exhibited by a radio wave returned from an ionosphere is next discussed in terms of two models in which the fading is assumed to be produced by movements of the diffracting centers in the ionosphere: in one of the models considered the irregularities are assumed to move with random velocities with mean velocity zero and in the other the irregular ionosphere is assumed to move overhead with a constant velocity, but so as to maintain its original form. The temporal auto-correlation function $\rho_R(\tau)$ of the amplitude of the irregularly fading signal is shown to be related to the velocity of the ionospheric diffracting centers. Thus, in the case when the diffracting centers are moving with velocities in the line of sight with a probability $p(v) \propto \exp(-v^2/2v_0^2)$, $\rho_R(\tau) \propto \exp(-16\pi^2 - v_0^2\tau^2/\lambda^2)$. For oblique reflexion v_0^2 in this last formula is effectively replaced by $v_0^2 \cos^2 i$ where i is the angle of incidence of the layer which acts like a plane reflector. *S. Chandrasekhar.*

Lengyel, Bela A. A note on reflection and transmission. J. Appl. Physics **22**, 263–264 (1951).

Important parameters which characterize wave propagation through the interface of two media or a transmission line containing an obstacle are the two reflection and the two transmission coefficients. For lossless discontinuities, the author shows that there are relations satisfied between the magnitudes of the reflection coefficients and phases of the reflection and transmission coefficients. Some applications are indicated. *A. E. Heins* (Pittsburgh, Pa.).

Lengyel, Bela A. Reflection and transmission at the surface of metal-plate media. J. Appl. Physics **22**, 265–276 (1951).

Carlson and the reviewer [Quart. Appl. Math. **4**, 313–329 (1947); **5**, 82–88 (1947); these Rev. **8**, 422, 614] discussed the effect of an electromagnetic plane wave incident upon an infinite set of parallel plates of zero thickness and perfect conductivity. Two cases were discussed there, the E and H plane cases for the case of specular reflection. The present paper treats the H plane case in the presence of one diffracted beam using these methods. In addition to supplying numerical data which were lacking in the first two papers, experimental results are also described. The E plane case in the presence of one diffracted beam has been treated by the reviewer [ibid. **8**, 281–291 (1950); these Rev. **12**, 375]. *A. E. Heins* (Pittsburgh, Pa.).

van Kampen, N. G. An asymptotic treatment of diffraction problems. II. Physica **16**, 817–821 (1950).

In a previous paper [Physica **14**, 575–590 (1948)] the author gave a general method for the expansion of an integral of the form $\int\int_R g(\xi, \eta)e^{ik/(\xi,\eta)}d\xi d\eta$ in descending powers of k. As the resulting formulae were rather complicated, he now gives an approximate treatment which leads directly to simplified results. *E. T. Copson* (St. Andrews).

Karp, Samuel N. Separation of variables and Wiener-Hopf techniques. New York University, Washington Square College, Mathematics Research Group, Research Rep. No. **EM-25**, ii+i+78 pp. (1950).

This report gives an interesting account of the various methods which are available for the solution of the boundary value problems of electrostatics or diffraction theory. After a critical introduction in § 1, the Green's function integral equation method is applied to the problem of diffraction by a straight edge. The integral equation of this problem, found first by Magnus, is of homogeneous Wiener-Hopf type; it

cannot be solved by elementary Fourier integral technique—it needs also the theory of functions of a complex valuable.

In § 3, the method of separation of variables is discussed, and it is pointed out that the nature of the boundary value problem depends on the coordinate system employed. For instance, the straight edge problem is a two-part mixed boundary value problem in Cartesian coordinates, yet it is a simple boundary value problem in polar coordinates. Or again, the problem of diffraction at a slit is a mixed three-part boundary value problem in Cartesian coordinates, but is a simple boundary value problem in elliptic coordinates. Naturally one pays the price of getting a simple boundary value problem in the complicated nature of the eigenfunctions.

The method of separation of variables applied to the straight edge problem in Cartesian coordinates leads to the representation of the solution in what Booker and Clemmow call an angular spectrum of plane waves. The two-part boundary value problem leads to dual integral equations, whose solution requires the Wiener-Hopf technique. The same method is applied in § 5 to the Carlson-Heins problem of diffraction by a staggered stack of parallel half-planes.

In § 6, the straight edge problem is discussed afresh in terms of polar coordinates. The solution is represented as an angular spectrum of cylindrical waves, and the boundary condition is now simple. The solution is completed without using complex function theory by means of a rather difficult transform theorem due to Kontorowitsch and Lebedev [Acad. Sci. USSR. J. Phys. 1, 229–241 (1939)]. Evidently the same transform theorem might have been applied to the Magnus integral equation. The remainder of the report deals with two electrostatic problems, in each of which the choice of a simple coordinate system makes it necessary to use the deeper Wiener-Hopf techniques. *E. T. Copson.*

Bonfiglioli, Guido. Proprietà di onde elettromagnetiche piane inomogenee. Alta Frequenza 19, 259–266 (1950).

In a homogeneous plane wave the electromagnetic field vectors are constant over the wave front; the author considers a "nonhomogeneous" plane wave in which the phase remains constant over the wave front but the amplitude may vary. A very elementary discussion of solutions of Maxwell's equations shows that a nonhomogeneous wave of this type can exist in an infinite homogeneous medium only if the medium has nonzero conductivity.

M. C. Gray (Murray Hill, N. J.).

Ott, Heinrich. Gibt es im Feld eines Senders eine Zenneckwelle? Arch. Elektr. Übertragung 5, 15–24 (1951).

Abstract based on the author's summary: New calculations verify the existence of a surface-wave term in the electromagnetic field of a vertical electric Hertzian dipole over a plane earth. This surface wave, which is present near the earth for large values of the refractive index of the earth only, differs by a factor ½ from Sommerfeld's original surface wave. It moreover constitutes part of a general "surface effect," which is of paramount importance for the attenuation of the surface wave and for the energy flow near the earth. The range of validity of Noether's solution [cf. R. Rothe, F. Ollendorff, and K. Pohlhausen, Funktionentheorie und ihre Anwendungen in der Technik, Springer, Berlin, 1931, part E, pp. 154–170] is narrowed, an error in Weyl's analysis [Ann. Physik (4) 60(365), 481–500 (1919)] is corrected, and errors in the arguments of Kahan and Eckart [C. R. Acad. Sci. Paris 226, 1513–1515 (1948); 227, 969–970 (1948); J. Phys. Radium (8) 10, 165–176 (1949); Physical Rev. (2) 76, 406–410 (1949); these Rev. 9, 637; 11, 143; see also Proc. I. R. E. 38, 807–812 (1950); these Rev. 12, 224] and Epstein [Proc. Nat. Acad. Sci. U. S. A. 33, 195–199 (1947); these Rev. 9, 126] are pointed out. [There is no discussion of the work done by W. H. Wise [Proc. I. R. E. 19, 1684–1689 (1931); Bull. Amer. Math. Soc. 41, 700–706 (1935); Physics 4, 354–358 (1933); Bell System Tech. J. 16, 35–44 (1937)] and S. O. Rice [ibid., 101–109 (1937)]. The reviewer's note cited by the author has now appeared [Physical Rev. (2) 80, 294 (1950); these Rev. 12, 376].]

C. J. Bouwkamp (Eindhoven).

Kahan, Théo, und Eckart, Gottfried. Die Nichtexistenz der Oberflächenwelle in der Dipolstrahlung über ebener Erde. Die Lösung einer alten Streitfrage. Arch. Elektr. Übertragung 5, 25–32 (1951).

This is the sixth of a series of papers on the surface-wave problem by the same authors [for references see the preceding review]. The first part of the paper (up to section 5.3) gives a fairly correct survey of the whole matter. [It may be pointed out, however, that the authors do not show where Sommerfeld failed in his original papers of 1909 and 1926, but they criticize him with regard to a modified version published in his lectures on theoretical physics [Partielle Differentialgleichungen der Physik, Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1947; these Rev. 10, 195]. There is a disturbing misprint on page 28, first column, last line, where "das Sternchen" should be replaced by "den Doppelkreis".] In the reviewer's opinion, the remainder of the authors' argument (in particular, sections 5.3 and 5.4) is not conclusive, since the same criticism expressed in earlier reviews does apply here [see also the reviewer's note in Physical Rev. (2) 80, 294 (1950); these Rev. 12, 376]. The paper concludes with an answer to Ott's criticism [see the preceding review]. *C. J. Bouwkamp* (Eindhoven).

Molžes, B. Ya. Concerning the excitation of electromagnetic oscillations. Akad. Nauk SSSR. Žurnal Tehn. Fiz. 20, 707–715 (1950). (Russian)

The author describes the use of scalar wave functions in solving Maxwell's equations in spherical and cylindrical coordinates. *E. N. Gilbert* (Murray Hill, N. J.).

Molžes, B. Ya. The computation for some grid structures. Akad. Nauk SSSR. Žurnal Tehn. Fiz. 20, 716–726 (1950). (Russian)

The reflection of plane electromagnetic waves by a plane array of thin closely spaced conducting wires is studied and the results obtained are used to get the reflection coefficient of such a grid as an obstacle in a wave guide. A spherical cavity resonator with walls consisting of closely spaced circular loops of wire is considered next and the attenuation constants of its normal modes are found. Finally a computation is made of the attenuation constant of waves in a cylindrical wave guide with grid wires for walls.

E. N. Gilbert (Murray Hill, N. J.).

Van Bladel, J. Expandability of a wave-guide field in terms of normal modes. J. Appl. Phys. 22, 68–69 (1951).

The author justifies the usual assumption that the electromagnetic field in a waveguide can be expanded in a linear combination of normal modes by considering the field (E, H) between two cross-sections S_1 and S_2 enclosing a known current distribution J, and comparing it with the field (E_p, H_p) which would exist in the cavity formed by metallizing S_1 and S_2. The latter field depends only on J and van-

ishes with J. Since the transverse distribution functions E_{zn} and H_{zn} for TM and TE modes form a closed complete set J can be expanded in terms of them and it can then be shown that the longitudinal field components can be expressed as linear combinations $E_z = E_{zp} + \sum C_n E_{zn}$, $H_z = H_{zp} + \sum D_n H_{zn}$. The proof is completed by showing that the transverse field components can also be expressed as sums of normal modes, using the same expansion coefficients C_n and D_n.

M. C. Gray (Murray Hill, N. J.).

Müller, Rolf. Über die Beugung von Rohrwellen an ebenen Blenden. Z. Naturforschung **5a**, 617–621 (1950).

The author first gives a general formulation of the diffraction problem for a perfectly conducting diaphragm in a waveguide of arbitrary cross-section, in terms of the boundary conditions over the closed and open portions of the diaphragm. Then he discusses the solution for a circular guide with a concentric diaphragm. It is shown that if the incident wave is circularly symmetric, of either TE or TM type, the diffracted wave includes only TE or TM modes respectively. For any other incident TE or TM mode, both mode types are present in the diffracted wave.

M. C. Gray (Murray Hill, N. J.).

Levine, Harold, and Papas, Charles H. Theory of the circular diffraction antenna. J. Appl. Phys. **22**, 29–43 (1951).

The antenna mentioned in the title consists of a semi-infinite coaxial wave guide fitted with an infinite plane conducting baffle. Its principal-mode behavior is described in terms of an equivalent circuit with reference to the open-end plane. Two different integral-equation formulations of the problem are recast into variational formulations which allow an accurate determination of the equivalent-circuit parameters. A comparison with experimental values is included.

C. J. Bouwkamp (Eindhoven).

Patrušev, V. L. The application of the method of curvilinear coordinates for the computation of Π-shaped endovibrators. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **20**, 727–734 (1950). (Russian)

The lowest mode electromagnetic field is found for a cavity in the form of a closed wave guide in which a partition is placed to make the interior Π-shaped. A conformal mapping is first found which solves a related static field problem; then a variational principle is used to find the lowest mode. *E. N. Gilbert* (Murray Hill, N. J.).

Gundlach, Friedrich Wilhelm. Zur Anwendung der Vierpoltheorie auf Hohlleitungssysteme. Arch. Elektr. Übertragung **4**, 342–348 (1950).

A waveguide transducer consists of an unknown internal system with two accessible "terminals", namely cross-sections of waveguides, analogous to the accessible terminals of an ordinary transducer. It is assumed that the system is linear and passive, that in each guide terminal only one mode (either TE or TM) is present, and that there is no dissipation. Then the transverse field components, E_t and H_t, at points P_1 and P_2 in each terminal satisfy transducer equations $E_{t1} = R_{11}H_{t1} - R_{12}H_{t2}$, $E_{t2} = R_{21}H_{t1} - R_{22}H_{t2}$, where the R's are constants. If the field components at one terminal can be measured, values of the transverse field resistance ($R = E_t/H_t$), the phase difference and the power output at the other terminal can be obtained from circle diagrams which are identical with the circle diagrams for the various voltage-current ratios in a network transducer.

M. C. Gray (Murray Hill, N. J.).

Roubine, Élie. Sur les ondes guidées par des circuits hélicoïdaux. C. R. Acad. Sci. Paris **232**, 1748–1750 (1951).

Friedman, Bernard. Amplification of the traveling wave tube. J. Appl. Phys. **22**, 443–447 (1951).

In a paper by Chu and Jackson [Proc. I. R. E. **36**, 853–863 (1948)] the theory of the helix-type traveling wave tube was developed, and a transcendental equation obtained for the propagation constants of the various modes. Approximate solutions of this equation were derived, but they had the drawback that the whole procedure had to be repeated for any change in value of any of the tube parameters. The present paper suggests an alternative approach, applying a perturbation method to the equation for the cold helix (that is, without the electron stream) which is more generally applicable. It is shown that the transcendental equation can be reduced to a cubic, which has one real and two complex roots under certain operating conditions. One of the complex roots corresponds to an amplified wave in the tube and the condition for maximum amplification is obtained.

M. C. Gray (Murray Hill, N. J.).

Ledinegg, E., und Urban, P. Über einige Ergebnisse aus der Theorie gekoppelter elektromagnetischer Hohlräume. Acta Physica Austriaca **4**, 180–196 (1950).

In an earlier paper [same Acta **2**, 198–213 (1948)] the authors derived a general formula for the natural frequencies of a system of weakly coupled cavity resonators. They now obtain explicit formulas for two resonator systems of interest in the centimeter wave-length range. The first consists of a circular waveguide divided into n resonator sections by means of equispaced transverse plates, the coupling between sections being supplied by small circular holes in each plate, symmetric about the axis. The second system is a set of cylindrical resonators coupled by concentric coaxial lines. The effect of dissipation is also discussed, and some experimental measurements are compared with the theoretical values. *M. C. Gray* (Murray Hill, N. J.).

Auluck, F. C., and Kothari, L. S. A note on Riesz potential. Proc. Cambridge Philos. Soc. **47**, 436–442 (1951).

The object of this paper is to discuss the representation of the Riesz electromagnetic vector-potential by a Fourier integral. For this purpose there is given a new definition of the electromagnetic vector-potential differing from that of Fremberg [Proc. Roy. Soc. London Ser. A. **188**, 18–31 (1946); these Rev. **8**, 302]. The new vector-potential generalises the Wentzel potential, whereas that of Fremberg generalises the classical Maxwell potential. The ideas developed in the note are applied to the problem of eliminating in a straightforward way the longitudinal part of the potential describing the electromagnetic field. *E. T. Copson.*

de Broglie, Louis. Énergie libre et fonction de Lagrange. Application à l'électrodynamique et à l'interaction entre courants et aimants permanents. Portugaliae Phys. **3**, 1–19 (1949).

The transformation relations connecting the Lagrange function $L(q, \dot{q})$ of mechanics with the thermodynamic free energy are discussed ($L = TS - E = -F$) and applied to a number of examples in electrodynamics. The examples include coupled circuits with and without the presence of permanent magnets. The result is an expression for the internal energy of the system such that one part of the expression is equal to the internal energy of the permanent

magnets and the other part is the energy of the mutual coupling. This result is used to obtain an expression for the internal energy of what is called an "ideal permanent magnet". There is some interesting discussion concerning the spin of an electron and the mutual energy between currents and magnets. *R. Truell* (Providence, R. I.).

Schlomka, Teodor. Zur Berechnung des elektromagnetischen Feldes bewegter Körper. Ann. Physik (6) **7**, 228–239 (1950).

Lundquist, S. Magneto-hydrostatic fields. Ark. Fys. **2**, 361–365 (1950).

The equation governing the hydrostatic equilibrium of a medium whose permeability μ is one and whose conductivity may be considered infinite is

$$\mu \operatorname{curl} \mathbf{H}\times\mathbf{H}=4\pi \operatorname{grad} p \tag{1}$$

where $\mathbf{H}$ denotes the magnetic field strength and p the hydrostatic pressure. An alternative form of this "magneto-hydrostatic equation" is

$$\mu(\mathbf{H}\cdot\operatorname{grad})\mathbf{H}=4\pi \operatorname{grad}(p+\tfrac{1}{2}\mu|\mathbf{H}|^2). \tag{2}$$

In the "force-free" case, when grad $p=0$, curl $\mathbf{H}\times\mathbf{H}=0$ and $\mathbf{H}$ must be of the form curl $\mathbf{H}=\alpha(\mathbf{r})\mathbf{H}$ where α is a function of position ($\mathbf{r}$). Since div $\mathbf{H}=0$, α must satisfy the equation grad $\alpha\cdot\mathbf{H}=0$. The author investigates if a force-free current distribution within a closed volume can give rise to a magnetic field outside of the volume and shows that a dipole field outside a sphere can result only from a toroidal current distribution within the sphere. When a pressure gradient exists, by resolving equation (2) along the normal and the binormal to a line of force, the author exhibits the well-known analogy (due to Alfvén) of the magnetic lines of force with elastic strings under a tension $|\mathbf{H}|^2/4\pi$ and acted upon by a pressure $(p+\frac{1}{2}\mu|\mathbf{H}|^2)/4\pi$. *S. Chandrasekhar.*

Åström, Ernst. On waves in an ionized gas. Ark. Fys. **2**, 443–457 (1951).

The equations governing the motion of ions in a medium of infinite compressibility and when no forces other than those arising from the macroscopic electric ($\mathbf{E}$) and magnetic ($\mathbf{H}$) fields, are $m_\nu d\mathbf{v}/dt=e_\nu(\mathbf{E}+\mathbf{v}\times\mathbf{B})$; $\mathbf{j}=\sum_\nu n_\nu e_\nu \mathbf{v}_\nu$ and $\partial\mathbf{D}/\partial t=\mathbf{j}+\epsilon_0\partial\mathbf{E}/\partial t$, where $\mathbf{v}$ denotes the velocity, $\mathbf{j}$ the current, n_ν the concentration of ions (of specific charge e_ν) of a certain species ν, $\mathbf{B}=\mu_0\mathbf{H}$, ϵ_0 the dielectric constant and μ_0 the magnetic permeability. It is shown that if these equations admit periodic solutions with a time dependent factor of the form $e^{i\omega t}$, then $\mathbf{D}=\epsilon_0\varepsilon\mathbf{E}$ where ε is a tensor whose nonvanishing components are given by

$$\epsilon_{1,2}=\epsilon_{xx}\pm i\epsilon_{xy}=1-\sum_\nu \Omega_\nu^2/\omega(\omega\mp\omega_\nu),$$

$$\epsilon_3=\epsilon_{zz}=1-\sum_\nu \Omega_\nu^2/\omega^2; \quad \epsilon_{xy}=-\epsilon_{yx} \text{ and } \epsilon_{xx}=\epsilon_{yy},$$

where $\omega_\nu=e_\nu|\mathbf{B}|/m_\nu$ and $\Omega_\nu=(n_\nu e_\nu^2/m_\nu\epsilon_0)^{\frac{1}{2}}$ are the gyro- and the plasma frequencies, respectively, of the ions of type ν. When the condition $\mathbf{D}=\epsilon_0\varepsilon\mathbf{E}$ is combined with Maxwell's other equations, it is found that $\mathbf{E}$ must satisfy the equation c^2 curl curl $\mathbf{E}=-\varepsilon\partial\mathbf{E}/\partial t$. This equation admits plane wave solutions of the form $\mathbf{E}=\text{constant}\times\exp i\omega(t-\mathbf{R}\cdot\mathbf{r}/\omega)$ where $\mathbf{R}$ is a unit vector inclined at an angle φ to the direction of $\mathbf{B}$, if

$$\tan^2\varphi=-\left[\frac{u^2}{c^2}-\frac{1}{\epsilon_1}\right]\left[\frac{u^2}{c^2}-\frac{1}{\epsilon_2}\right]\Big/\left[\frac{u^2}{c^2}-\frac{1}{\epsilon_3}\right]\left[\frac{u^2}{c^2}-\frac{1}{2}\left(\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}\right)\right].$$

Accordingly, we have two types of solutions; the author calls them the "ordinary wave" and the "extraordinary wave." The cases when the wave normal $\mathbf{R}$ is in the direction of $\mathbf{B}$, and at right angles to the direction of $\mathbf{B}$, are considered in some detail. The case when the coefficients in the dielectric tensor can be expanded in a power series in ω is also examined and it is shown how the usual type of magneto-hydrodynamic waves appear as the limit of the ordinary waves. *S. Chandrasekhar* (Williams Bay, Wis.).

Knödel, Walter. Über Zerfällungen. Monatsh. Math. **55**, 20–27 (1951).

The author first derives MacMahon's generating identity [Electrician **28**, 601–602 (1892)] for the number of two-terminal series-parallel networks of unlabelled abstract elements. This result, which MacMahon gave without proof, has also been proved by Riordan and Shannon [J. Math. Physics **21**, 83–93 (1942); these Rev. **4**, 151]. Then he gives the following generating identity for labelled networks:

$$1+x+\sum A_n x^n/n!=\exp(a_1 x+a_2 x^2/2!+\cdots)$$

with A_n the number of networks with n branches, and $a_n=\frac{1}{2}A_n$ the number of essentially series (or essentially parallel) networks. *J. Riordan* (New York, N. Y.).

Gavrilov, M. A. The structural classification of relay-contact schemes. Avtomatika i Telemehanika **8**, 297–307 (1947). (Russian)

Relay networks are classified into eleven different types. The type depends on the possible behavior of the network (whether or not the network can be made to chatter, lock in some of its relays, etc.) and can be deduced from a circuit diagram with the aid of certain structural matrices.
E. N. Gilbert (Murray Hill, N. J.).

Gavrilov, M. A., and Hvoščuk, V. A. The method of partial inversion in relay schemes. Doklady Akad. Nauk SSSR (N.S.) **75**, 685–687 (1950). (Russian)

Two examples are given in which planar relay circuits may be transformed into equivalent circuits by the following process: (1) the dual network is drawn, and (2) certain parts of the dual network are replaced by their duals. The proper way to choose the parts to be dualized in (2) seems to depend on guesswork. *E. N. Gilbert* (Murray Hill, N. J.).

Ionkin, P. A. On substitution schemes for passive networks. Èlektričestvo **1950**, no. 11, 74–78 (1950). (Russian)

The equations of an N terminal linear network expressing the currents entering the nodes in terms of the node potentials are transformed to express the currents in terms of the potential differences between pairs of nodes.
E. N. Gilbert (Murray Hill, N. J.).

González Domínguez, Alberto. On some points of the mathematical theory of linear circuits. Revista Unión Mat. Argentina **14**, 275–322 (1950). (Spanish. English summary)

Much of the mathematical theory of linear circuits, as it is found in present day papers and treatises, is based upon the assumption that the circuits consist of finite collections of lumped elements. On the other hand, many of the relations obtained in the theory hold under much broader conditions. Thus there has existed some uncertainty as to just what assumptions are required for the deeper parts of the theory. The chief purpose of this paper is to clarify this

point. As the fundamental function characterizing a circuit, the author takes the response $G(t)$ to the excitation which is 0 or 1 according as $t<0$ or $t>0$. It is assumed that $G(t)$ is 0 for $t<0$, is of bounded variation in each finite interval, and has no singular part. The function of the complex frequency p characterizing the circuit (i.e. the transfer ratio) is defined by the equation $g(p)=\int_0^\infty e^{-pt}dG(t)$. A stable circuit is defined to be one such that the responses to bounded excitations are bounded, and such a circuit is characterized by the fact that $G(t)$ is of bounded total variation over the interval $(0, \infty)$. Various familiar properties of stable circuits are obtained from this point of view. The first part of the paper closes with some consideration of the properties of circuits for which the functions $G(t)$ are monotone increasing.

The second part is devoted to the study of what the author calls Wiener transfer ratios. These are the functions $g(p)$ corresponding to functions $G(t)$ such that $[G'(t)]^2$ is summable over the interval $(0, \infty)$. A canonical representation of Wiener transfer ratios is given, and used to obtain various properties of the ratios. In particular, the canonical representation shows that the most general Wiener transfer ratio is the product of the transfer ratio of what is usually called a "minimum phase circuit" and the transfer ratio of a generalization of what is usually called an "all-pass circuit." It is pointed out that if $g_0(p)$ is the transfer ratio of a minimum phase circuit, and if $g(p)$ is a Wiener transfer ratio such that $|g_0(p)|=|g(p)|$ for all pure imaginary values of p, then we have $|g_0(p)|\geqq|g(p)|$ for any p with a positive real part. Accordingly, the author proposes to call minimum phase circuits by the (perhaps better) name "maximum modulus circuits." The second part of the paper terminates with a rather extensive discussion of the relations which exist between the real and imaginary parts of $g(p)$, and of $\log g(p)$, for pure imaginary values of p. Here there are found some extensions and generalizations of previously known results.

In the third part the author discusses some problems relating to the synthesis of circuits having prescribed properties. In particular, he describes a method for the synthesis of a circuit such that $|g(i\omega)|$ is a prescribed function of the real variable ω. This method differs from that of Wiener and Lee in that the required $g(p)$ is obtained as a product, rather than as a sum, of elemental transfer ratios.

L. A. MacColl (New York, N. Y.).

Lunc, A. G. Synthesis and analysis of relay-contact schemes by means of characteristic functions. Doklady Akad. Nauk SSSR (N.S.) **75**, 201–204 (1950). (Russian)

To each of the nodes $1, 2, \cdots, N$ of a switching network a variable $x_1, x_2, \cdots, x_N$ is associated; these variables assume the values 0, 1 and each x_i has an inverse x_i' which has the opposite value from x_i. The characteristic function of the network is a quadratic form $\sum a_{ij}x_ix_j'$ where a_{ij} is the Boolean variable which has the value 1 when the branch joining nodes i and j is closed and 0 otherwise. The Boolean function which describes when there is a closed path in the network joining terminals 1 and 2 may be found as the coefficient of $x_1x_2'+x_1'x_2$ in the expression obtained by multiplying together the 2^{N-2} terms obtained from the characteristic function by assigning numerical values to $x_1, \cdots, x_N$ in all possible ways.

E. N. Gilbert (Murray Hill, N. J.).

Ming, Nai-Ta. Verwirklichung von linearen Vierpolschaltungen vorgeschriebener Frequenzabhängigkeit unter Berücksichtigung gleicher Spulenverluste und gleicher Kondensatorenverluste. Arch. Elektrotechnik **39**, 496–507 (1949).

The problem treated is a special case of the synthesis of two-terminal pair networks with lossy elements. An attempt is made to realize the transfer function of a two-terminal pair network constituted with lossy inductances and capacitances, with no inductive coupling. The procedure followed is to determine the matrix elements of the corresponding lossless reactance network with the object of realizing the prescribed transfer function (to within a constant factor) when inductances and capacitances are replaced by lossy inductances and capacitances. The results are inconclusive since the complete conditions for realizability are not obtained. A discussion of these conditions and some numerical examples are included. *R. Kahal* (St. Louis, Mo.).

Ming, Nai-Ta. Die linearen Wechselstromschaltungen unter Berueecksichtigung der Verluste. Eng. Rep. Nat. Tsing Hua Univ. **3**, no. 2, 118–134 (1948).

This paper is a summary of results on the synthesis of the driving point impedance of linear networks with lossy elements which later appeared elsewhere [Sci. Rep. Nat. Tsing Hua Univ. Ser. A. **5**, 350–377 (1949); Arch. Elektrotechnik **39**, 359–387 (1949); these Rev. **12**, 148].

R. Kahal (St. Louis, Mo.).

Ming, Nai-ta. Network of constant resistance as driving point impedance under prescribed frequency characteristic. Eng. Rep. Nat. Tsing Hua Univ. **4**, no. 2, 11–49 (1950). (Chinese. English summary)

"A new general method to design a constant resistance network under prescribed frequency characteristic such as transfer, impedance, or admittance, which is separated into elementary factors is presented. A network corresponding to each factor is realized, principally according to Cauer [Theorie der linearen Wechselstromschaltungen, Akademische Verlagsgesellschaft, Leipzig, 1941], which is mainly designed for combination of two filters and the whole network consists of all the corresponding individual networks in cascade connection."

From the author's summary.

Zadeh, Lotfi A. Circuit analysis of linear varying-parameter networks. J. Appl. Phys. **21**, 1171–1177 (1950).

This paper gives some results of the investigation of electrical networks constructed with elements which are linear with current and voltage but otherwise time variable. Such a network may be termed a variable network. The investigations are made in the frequency domain, and the familiar techniques and procedures used for the analysis of fixed linear networks are suitably modified and extended. It is shown that the behaviour of a variable network can be expressed in terms of a linear time-dependent operator $H(p; t)$, the system function. Instead of directly attacking the problem of finding the system function the emphasis is, rather, on the development of rules for combining system functions of various network combinations. Thus, by analogy with fixed linear networks, the system function of a variable network may then be expressed in terms of the system functions of the constituent elements. Accordingly, the basic concepts of impedance, admittance, gain, etc., their important properties, and the relationships between them are discussed and the modification of fundamental network properties is indicated. *R. Kahal.*

Mathews, W. E. Transmission-line equivalent of electronic traveling-wave systems. J. Appl. Physics 22, 310–316 (1951).

The problem of coupling between electron beams and electromagnetic fields is investigated from the viewpoint of equivalent transmission line systems. The analysis is restricted to small signals, electron motion in the longitudinal direction, and propagation modes varying only in one normal coordinate. A lumped network representation of a unit longitudinal length (small compared with a wave length) is derived, the elements of the network representing admittances per unit width of the longitudinal element. This equivalent circuit describes the interaction or coupling between the two elements (beam and propagating circuit). With electron beam and propagating circuit separately regarded as transmission lines it is shown that the system is equivalent to two coupled distributed-parameter transmission lines moving relative to each other. Methods for calculating the parameters in certain special cases is given.

R. Kahal (St. Louis, Mo.).

Ahmed, Rais. The nonlinear method of circuit analysis. Proc. Nat. Inst. Sci. India 16, 255–262 (1950).

The discontinuous method of Mandelstein and Papelexi is discussed. *N. Levinson* (Cambridge, Mass.).

Quantum Mechanics

Bodiou, Georges. Recherches sur les fondements du calcul quantique des probabilités dans les cas purs (Espace de Hilbert. Principe d'ondulisation). Ann. Physique (12) 5, 451–536 (1950).

The author believes that the usual approaches to the foundations of quantum mechanics are unsatisfactory because Hilbert space is introduced first, while the probability relations appear as secondary and incidental. He proposes to remedy this and found quantum mechanics on probability. Among others, two difficulties have to be overcome: (1) the inconsistency of the classical particle and laws of probability with quantum mechanics (giving, e.g., interference in the experiment of Young's two slits); and (2) the artificiality of the conventional process of quantizing the Hamiltonian by replacing q, p by the operators of x-multiplication and $(h/2\pi i)d/dx$.

The author attempts to overcome (1) by modifying the classical notion of particle ("that solidarity between position and momentum persisting in time") by means of a statistical concept ("stochastic corpuscular weight"): In Young's experiment, let p_1 be the a posteriori probability, computed by Bayes' formula, that "a particle passed through slit no. 1"; form an urn whose proportion of black balls is p_1; draw a ball at random; if it is black report that "a particle passed through slit no. 1"; if non-black report that "no particle passed through slit no. 1." The same process is applied to slit number 2. Thus a random number (0, 1, 2; expected number: 1) of particles is reported, with a "precision" expressible in terms of its standard deviation. From now on this "statistical particle" will replace the classical one in physical reasoning; and it shall be a fundamental principle that in all quantum reasoning and assignment of probabilities, classical addition of probabilities must hold to within the limits of the precision mentioned above ("Postulate (Q)"). This procedure is regarded as analogous to the replacement of metaphysical simultaneity by the operational definition of simultaneity given in restricted relativity.

To overcome (2), the author introduces his Postulate (G) of preservation of the normal distribution: Granting that the relation between the q- and p-representations is given by $(p|)=\int_{-\infty}^{+\infty} f(p, q)(q|)dq$, the kernel $f(p, q)$ must be such that if $|(q|)|^2$ is a Gaussian density, so is $|(p|)|^2$; and vice versa. This (with subsidiary assumptions) leads to an $f(p, q)$ making the above relation essentially a Fourier transformation. Whence the quantization of the Hamiltonian, the indeterminacy relations, etc., are derived, following a method of Tolman.

Various further developments are given, concerning complex correlations and special distributions. Much of conventional quantum mechanics is subjected to a critique. In trying to carry out his program the author takes over so many parts of the conventional theory that it is not clear to the reviewer, on a first reading of the paper, just what a pure development of the whole theory from the author's point of view would look like. *B. O. Koopman.*

Morette, Cécile. On the definition and approximation of Feynman's path integrals. Physical Rev. (2) 81, 848–852 (1951).

R. P. Feynman [Rev. Modern Physics 20, 367–387 (1948); these Rev. 10, 224] has developed a Lagrangian formulation of quantum mechanics, based upon the following single postulate. The probability amplitude for a particle to go from a space-time point x^A to a space-time point x^B is given by

$$K(x^B, x^A)=\int \exp(iS[x]/\hbar)d(\text{paths}),$$

the integral being extended over all paths $x(\tau)$ from x^A to x^B. Here $x(\tau)$ is the parametric representation of a world-line, τ is the proper time, and

$$S[x]=\int_{\tau^A}^{\tau^B} L[x, (dx/d\tau)]d\tau$$

is the action-integral along the path $x(\tau)$.

The mathematical imprecision of the notion of functional integration involved in the definition of $K(x^B, x^A)$ has been a serious obstacle to the use of the Feynman formalism, which is in other respects considerably simpler and more elegant than the usual Hamiltonian formulation of quantum theory. The present paper is concerned with two problems: (i) to make the definition of $K(x^B, x^A)$ precise, and (ii) to obtain a general approximate formula for $K(x^B, x^A)$ which is in error by terms of order $\hbar$ in the classical limit when $\hbar \to 0$. Problem (i) is dealt with by dividing the world-line $x(\tau)$ into a finite number n of intervals, replacing the integral $S[x]$ by a finite sum of n terms, and replacing the functional integral $K(x^B, x^A)$ by an ordinary $(4n-4)$-fold integral; finally, the integer n is supposed to tend to infinity. Problem (ii) is dealt with by making a formal Taylor series expansion of $S[x]$ in powers of $(x-\bar{x})$, where $\bar{x}(\tau)$ is the classical orbit of a particle travelling from x^A to x^B.

The results of this paper will satisfy the needs of a physicist who wishes to use the Feynman formalism in practical physical situations. They will not satisfy a mathematician who looks for a rigorous theoretical basis for Feynman's methods. In particular, there is no discussion of the mathematically crucial question, whether the multiple integrals in fact tend to a limit as $n \to \infty$. Certainly this will

not be true without some strong restrictions on the form of the Lagrangian. *F. J. Dyson* (Ithaca, N. Y.).

Kato, Tosio. Fundamental properties of Hamiltonian operators of Schrödinger type. Trans. Amer. Math. Soc. **70**, 195–211 (1951).

It is shown that the Hamiltonian for a finite set of nonrelativistic quantum-theoretical particles with interaction determined by a suitable potential energy (e.g. of Coulomb type) is hypermaximal symmetric (or self-adjoint in the strict Hilbert space sense); the energy levels of the system therefore exist and are unique. This was known previously in some special cases, e.g. that of the hydrogen atom, via more or less explicit determination of the spectral resolution of the operator. The present method establishes the character of the operator by Hilbert space techniques not involving the determination of the spectral resolution.

Let S be the direct product of n 3-dimensional Euclidean spaces, with general element $(\mathfrak{r}_1, \cdots, \mathfrak{r}_n)$, and let H be the differential operator on functions on S given by the equation $H=-\sum_{i=1}^{n}\mu_i \operatorname{grad}_i^2-\mu_0(\sum_{i=1}^{n}\operatorname{grad}_i)^2+V(\mathfrak{r}_1, \cdots, \mathfrak{r}_n)$, where the μ_i are nonnegative constants, the μ_i with $i\geqq 1$ being positive. Let H_1 be the operator on $L_2(S)$ obtained by allowing H to act on the domain D_1 of all functions of the form $p(\mathfrak{r}_1, \cdots, \mathfrak{r}_n)\exp\{-\frac{1}{2}\sum_{i=1}^{n}\mathfrak{r}_i^2\}$, where p is a polynomial in the coordinates of the $\mathfrak{r}_i$. It is assumed that the (real) potential has the form

$$V(\mathfrak{r}_1, \cdots, \mathfrak{r}_n)=W(\mathfrak{r}_1, \cdots, \mathfrak{r}_n)+\sum_{i=1}^{n}U_{0i}(\mathfrak{r}_i)+\sum_{i<j}U_{ij}(\mathfrak{r}_i-\mathfrak{r}_j),$$

where W is bounded measurable, and the U_{ij} are square-integrable on every bounded set and bounded near ∞. The main result is then that H_1 is essentially hypermaximal symmetric: $H_1^*=H_1^{**}$, or the closure $\bar{H}$ of H_1 is hypermaximal. Moreover, the domain of $\bar{H}$ is independent of the potential (and can be evaluated by taking the Fourier transform of H_1 in the case of zero potential), and every eigenfunction of $\bar{H}$ is [by virtue of results of Friedrichs, Amer. J. Math. 61, 523–544 (1939)] twice differentiable and satisfies the Schrödinger wave equation in every region in which the potential function has a sufficient number of derivatives. An allowable type of potential is an inverse distance to a power $<\frac{3}{2}$; the necessity of a restriction on the potential is shown by the author's statement that $H_1^*\neq H_1^{**}$ in the case of an inverse square potential.

The proof first examines the kinetic energy operator $-\sum_{i=1}^{n}\mu_i \operatorname{grad}_i^2-\mu_0(\sum_{i=1}^{n}\operatorname{grad}_i)^2$ (on D_1) whose closure is readily calculated and seen to be hypermaximal on a certain domain D_0. Next an inequality is derived which shows that D_0 is in the domain of the operation of multiplication by the potential energy part of the Hamiltonian, and a second use of this inequality shows the hypermaximality of the sum $\bar{H}$. A number of cognate topics are examined in the light of these results, and in particular the validity of the Rayleigh-Ritz method in certain situations follows.

I. E. Segal (Chicago, Ill.).

Kato, Tosio. On the existence of solutions of the helium wave equation. Trans. Amer. Math. Soc. **70**, 212–218 (1951).

The author proves that the conventional helium wave equation admits solutions, and in particular establishes the existence of the ground state solution; previous authors, who relied on series expansions and other non-Hilbert space methods, had not found it possible even to show rigorously the existence of the latter solution. The main tool in the proof is the spectral theorem for a hypermaximal symmetric operator; this theorem is applicable to the present situation because the Hamiltonian for a many-particle nonrelativistic system with suitable potential function was shown earlier by the author to be hypermaximal symmetric [see the preceding review]. The number of linearly independent eigenfunctions (or solutions of the wave equation) is estimated as at least 25585 with the conventional physical constants and infinite with the nuclear mass taken as ∞. The proof relies on a comparison of the pertinent Hamiltonian $\bar{H}$ to the Hamiltonian H' of a pair of noninteracting particles bound to a nucleus by Coulomb potentials, the nature of the latter Hamiltonian being clear from the theory of the hydrogen atom. It is shown that $\bar{H}\geqq H'$, from which it follows that $\dim E(\lambda)\leqq\dim E'(\lambda)$ for real λ, where $\bar{H}=\int\lambda dE(\lambda)$ and $H'=\int\lambda dE'(\lambda)$, and $\dim P$ denotes the dimension of the range of P. A sharper comparison of the same general nature discloses the stated result, with the eigenfunctions in question constituting a basis for the range of $E(\lambda)$, for an appropriate λ. *I. E. Segal* (Chicago, Ill.).

Rideau, Guy. Sur la quatrième relation d'incertitude. C. R. Acad. Sci. Paris **232**, 2007–2009 (1951).

Faure, Robert. Intégrale première du premier ordre dépendant du temps. Étude de deux cas particuliers. Signification des matrices $\alpha_1, \alpha_2, \alpha_3$, l'opérateur $(h/2\pi i)(\partial/\partial t)$ joue un rôle analogue à ceux des opérateurs $(h/2\pi i)(\partial/\partial q_i)$ C. R. Acad. Sci. Paris **232**, 1738–1740 (1951).

***Corson, E. M. Perturbation Methods in the Quantum Mechanics of n-Electron Systems.** Hafner Publishing Co., New York, N. Y., 1950. xi+308 pp. $11.00.

This effort to survey the methods available for the study of many-particle systems in quantum mechanics is most timely and useful. Its scope is largely restricted to systems of identical particles of spin $\frac{1}{2}$. The first five chapters introduce the basic mathematical and physical ideas needed in Dirac's approach to quantum mechanics, viz., linear operators, eigenvalues, observables, transformation theory, perturbation theory. The treatment improves on the usual ones in three particulars: an elegant method using the shift operators of Schrödinger is employed to obtain all the basic formulas related to the angular momentum operators; the meaning of the Schrödinger and Heisenberg pictures is explained with great care; Dirac's density matrix is discussed.

Chapter VI discusses finite groups and their representations including the orthogonality conditions. Emphasis is laid on the permutation group since it is the only group which is seriously applied in the sequel. Indeed, the main function of this chapter seems to be "to eliminate any seeming mystery from the physical approach developed by Dirac" (p. 93) and to enable the simple Dirac vector model to be generalized. [Reviewer's comment: This seems to mark the definitive triumph of group theory since it used to be claimed as an advantage of Dirac's methods that they avoided group theory, but here it is recognized that a knowledge of groups is almost essential if they are to be understood!]. Chapter VII discusses the exclusion principle and continues the presentation of the density matrix.

The next three chapters constitute the central core of the book. Chapter VIII formulates the variational principle and immediately applies it to derive the Hartree and Fock equations. A concluding section presents Møller and Plesset's perturbation treatment of the limitations of these equations.

Chapter IX develops the Thomas-Fermi statistical method, giving a careful discussion of boundary conditions. Chapter X, seventy pages in length, presents the Dirac vector model. The simple vector model is applied to calculate the term values for six different configurations of which the most complicated is $d^3(^4F^4P^2H^2G^2F^2D^2D^2P)$. In order to extend the method, matrices for the transpositions in certain irreducible representations of the symmetric group S_n are required. Two methods of finding these are given of which the simpler is due to Yamanouchi [Proc. Phys.-Math. Soc. Japan (3) 18, 623–640 (1936)]. The generalization of the vector model due to Serber, which uses the representations of S_n is explained and applied to separate the energies of the two ^2D terms in the above configuration. Finally, the power of the vector model is illustrated by application to some molecular problems: the exchange energy of CH_4, the activation energy for a system of four electrons in nonidentical orbits, the factorization of the secular equation for a six electron problem with hexagonal structure such as C_6H_6, and an eight electron problem with cubic structure. Chapter XI gives a detailed discussion of second quantization which is used to develop alternative proofs of the Hartree and Fock equations and the basic formula of the vector method. The last chapter gives a brief introduction to the Heisenberg S matrix.

Reviewer's comments: Undoubtedly, chapter X is the most useful part of the book for, as far as the reviewer is aware, its presentation of the Dirac vector model is unequalled in thoroughness or clarity by any other account at present available. On the other hand, the chapter on second quantization seems rather more involved than necessary, possibly due to a superfluity of bras and kets. There is considerable nuisance value in the 415 footnotes of which at least half are integral to, and should have been incorporated in, the text. There are no serious misprints and the typography is so excellent as to well serve as a model for other printers of mathematical texts. It is remarkable that there is little overlap between this book and the one of Gombas [Theorie und Lösungmethoden des Mehrteilchenproblems der Wellenmechanik, Birkhäuser, Basel, 1950; these Rev. 12, 378] on the same subject. Rather, they complement each other with Corson emphasizing basic general theory and Gombas practical methods of obtaining numerical results. *A. J. Coleman* (Toronto, Ont.).

Roca, Marcelo Alonso. Schrödinger's theory of dispersion by Born's method of perturbations. Revista Soc. Cubana Ci. Fís. Mat. 2, 136–141 (1949). (Spanish)

The author derives the formulas obtained by Schrödinger for the dispersion of an electromagnetic wave by a quantum mechanical system, by using the method of time-dependent perturbations introduced by Born. *O. Frink.*

Curtiss, C. F., Hirschfelder, J. O., and Adler, F. T. The separation of the rotational coordinates from the N-particle Schroedinger equation. J. Chem. Phys. 18, 1638–1642 (1950).

Let ψ_μ^L be the Schroedinger wave function for a system of N particles interacting with one another in the absence of any external field, L and μ being the quantum numbers representing the total angular momentum and its component along the Z-axis, respectively. On the basis of group theory one can show that this can be written $\psi_\nu^L=\sum_\mu D^L(R)^*_{\mu\nu}\chi_\mu^L$, where the $D^L(R)_{\mu\nu}$ are the representation coefficients for the Lth irreducible representation of the three-dimensional rotation group and are functions of the three coordinates describing the orientation of the particle system, while the χ_ν^L are functions of the $3N-6$ coordinates specifying the relative configuration of the system. From the Schroedinger equation for ψ_ν^L, a set of coupled differential equations is obtained for χ_ν^L. The special case $N=3$ is discussed in detail. The present paper is related to an earlier one by Hirschfelder and Wigner [Proc. Nat. Acad. Sci. U. S. A. 21, 113–119 (1935)], but differs in that it makes use of variables among which no relations exist. *N. Rosen.*

Costa de Beauregard, Olivier. Développement d'une théorie de Marcel Riesz. Forme covariante de la fonction de distribution de l'impulsion-énergie de l'électron libre. C. R. Acad. Sci. Paris 232, 804–806 (1951).

The Fourier decomposition of a solution of the Dirac equation for a free electron is written in an invariantive form. This equation is also inverted. The results are interpreted in terms of the current vector. *A. H. Taub.*

Géhéniau, J. Les fonctions singulières de l'équation de Klein-Gordon, tenant compte d'un champ magnétique extérieur. Physica 16, 822–830 (1950).

Exact singular solutions of the relativistic Schrödinger equation are derived for a particle of charge e and rest mass m in a uniform and constant magnetic field.

A. H. Taub (Urbana, Ill.).

Géhéniau, J., et Servranckx, R. La polarisabilité du proton. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 37, 138–145 (1951).

The polarizability of the proton is investigated on the basis of pseudo-scalar meson theory. To do this the energy of a single proton in a homogeneous field is calculated to terms in the square of the field strength. As is to be expected in the general theory of renormalization, a mass renormalization but no charge renormalization is found. After removal of the infinite mass term, a finite result is obtained. This proves completely negligible at field strengths obtainable in the laboratory. *J. M. Luttinger.*

Iwata, Giiti. La théorie du champ unitaire et la théorie quantique. J. Phys. Soc. Japan 2, 21–24 (1949).

The author compares the equations of a geodesic in Veblen's formulation of projective relativity with the equations of motion of a charged particle in an electromagnetic field in general relativity. He finds that these equations will be identical if one interprets one constant of the motion in projective relativity as the charge. The mass of the particle then decreases with the charge on the particle. By restricting the dependence of a wave function on the fifth variable of projective relativity the author then obtains a relation which the measure of charge must satisfy. *A. H. Taub.*

Iwata, Giti. Gauge-spin transformations and wave equations. Progress Theoret. Physics 5, 983–992 (1950).

The author considers wave equations of the form $\gamma^i\partial\psi/\partial x^i+F\psi=0$ where the γ^i are a set of $2^n\times 2^n$ Dirac matrices and F is an arbitrary matrix which may be determined by a set of tensor fields, the coefficients of the expansion of F in terms of the basis generated by the γ^i. An attempt is made to relate the transformation law of these tensor fields with general spin transformations and to interpret the latter as gauge transformations of the tensor fields. Arguments are given for treating the full spin group instead of the unitary one. *A. H. Taub* (Urbana, Ill.).

Gião, Antonio. Équations du champ, équations du mouvement et fonctions d'onde. I. J. Phys. Radium (8) **12**, 31–40 (1951).

The structure of a Riemannian V_n imbedded in a V_{n+1} is determined by its "internal" and "external" metrics which are generalizations of the familiar first and second differential quadratic forms of Gauss for a surface in three-space. Assuming that the universe consists of a "container" describable by such a V_n and "contents" describable by appropriate tensors, the author deduces that the container of an autonomous self-determined universe must be a four-dimensional space of signature $(+, +, +, -)$ imbedded in a Euclidean five-space. The internal metric is interpreted, as in general relativity, as the gravitational potentials; the external metric is related to the electromagnetic field. Equations of motion and of conservation are deduced. Spin forces are introduced by approximating elementary particles by rotating spheres. *A. J. Coleman* (Toronto, Ont.).

Gião, Antonio. Équations du champ, équations du mouvement et fonctions d'onde. II. J. Phys. Radium (8) **12**, 99–106 (1951).

The systematic presentation of the author's unitary field theory, begun in the article reviewed above, is continued. Wave functions are introduced and the internal and external metric fields are quantized. For a single particle new nuclear and electromagnetic potentials are deduced which coincide with those recently proposed by de Broglie [C. R. Acad. Sci. Paris **229**, 157–161, 269–271, 401–404 (1949); Physical Rev. (2) **76**, 862–863 (1949); these Rev. **11**, 302] and which lead to finite self-energies. Applied to the hydrogen spectrum the theory leads to the conclusion that the hyperfine structure arises from a small perturbation of the electron self-energy by interaction with the non-Coulomb proton field. *A. J. Coleman* (Toronto, Ont.).

Votruba, Václav. À propos de la covariance d'équations de champs physiques par rapport au groupe de Lorentz. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **47** (1946), 79–90 (1950).

Spinor analysis is used to study the transformation properties of the "Maxwell-like" equations of Frenkel and Madelung [J. Frenkel, Wave Mechanics. Advanced General Theory, Oxford, 1934, p. 264]. The author concludes that the equivalence of these equations with those of Dirac, alleged by Frenkel, can be maintained for a free electron but fails for a bound electron. *A. J. Coleman.*

Votruba, Václav. Pair production by gamma-rays in the field of an electron. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **49** (1948), 19–49 (1950).

The cross-section for the production of an electron-positron pair in the presence of an electron is computed by conventional perturbation theory. The rather complicated differential cross-section is found, taking into account hole theory. Approximate total cross-sections are obtained in the vicinity of the threshhold for the process and in the limiting case of an extremely high energy photon. *K. M. Case.*

Nishijima, Kazuhiko. On the generalized transformation function and the integrability condition. Progress Theoret. Physics **5**, 813–821 (1950).

Koba has proposed an integral definition of the transformation function $U(\sigma, \sigma_0)$ directly in terms of the interaction Lagrangian [Progress Theoret. Physics **5**, 696–717 (1950); these Rev. **12**, 464]. It is here shown that there is always a Hamiltonian density related to $U(\sigma, \sigma_0)$ which satisfies the integrability conditions and the Tomanaga-Schwinger equation. An analogue of the Dirac-von Neumann density operator is defined and its simplest properties derived. An appendix discusses difficulties in the present theory and comes to the "very gloomy conclusion" that results derivable from a variational principle in more orthodox theories are no longer immediately valid. In particular it is not evident how the condition of gauge invariance, concerned with the total Lagrangian, is to be imposed on this theory which is formulated wholly in terms of the interaction Lagrangian. *A. J. Coleman* (Toronto, Ont.).

Ludwig, Günther. Ansatz zu einer divergenzfreien Quantenelektrodynamik. Z. Naturforschung **5a**, 637–641 (1950).

In the first part of the paper it is shown that the divergences of field theory are not due to perturbation theory but rather to the structure of the commonly introduced interaction terms. This is done by showing that there is no everywhere dense set of vectors of Hilbert space for which the interaction operator is defined. The second portion of the paper proposes a quantum electrodynamics without electron self energy effects. This is achieved by eliminating the electromagnetic field by expressing it in terms of the electron current operator. The resulting theory is then strikingly similar to the absorber theory of radiation [J. A. Wheeler and R. P. Feynman, Rev. Modern Physics **17**, 157–181 (1945)]. In the case of quantization without hole theory an expression for the energy momentum four-vector is set up and it is shown that its expectation value is the same as with no interaction. For a hole theory it is indicated that convergence can be achieved by means of regulators. Since emission can only take place accompanied by absorption the negative energy difficulties pointed out by other authors [e.g., Pais and Uhlenbeck, Physical Rev. (2) **79**, 145–165 (1950); these Rev. **12**, 227] will not occur. *K. M. Case.*

Kümmel, Hermann. Der Energie-Impulsvektor in der unitären divergenzfreien Quantenelektrodynamik. Z. Naturforschung **5a**, 642–644 (1950).

An expression for the energy-momentum four-vector appropriate to the unitary non-hole theory electrodynamics proposed by Ludwig [see the preceding review] is obtained. This is found by starting from the four-vector constructed by Wheeler and Feynman [Rev. Modern Physics **21**, 425–433 (1949); these Rev. **11**, 293] for classical particles. This is then generalized to the case of a classical electron field. Terms are then reordered so as to be appropriate to a quantized electron field. *K. M. Case.*

Sakata, Shoichi, and Umezawa, Hiroomi. On the applicability of the method of the mixed fields in the theory of the elementary particles. Progress Theoret. Physics **5**, 682–691 (1950).

This is a review article discussing the attempts that have been made by various authors to overcome the divergence difficulties of quantized field theories by introducing additional hypothetical fields. The idea is to achieve a compensation of divergences between the various fields. However, a complete compensation of this kind is not in practice obtainable [D. Feldman, Physical Rev. (2) **76**, 1369–1375 (1949); these Rev. **12**, 150]. The authors suggest that the compensation program may still have an important part to play in the theory of elementary particles, although it has

not achieved the success which at one time was hoped from it. *F. J. Dyson* (Ithaca, N. Y.).

Tenaglia, Livio. Osservazioni sulle divergenze dell'elettromagnetismo. Atti Relaz. Accad. Pugliese Sci. N.S. **7**, Parte I, 187–190 (1949).

Rather general remarks, suggested by Feynman's cut-off function, leading to the conclusion "the solution of the divergence problem in electromagnetic field theory will only be achieved by the creation of a formalism which takes account of the reaction of the particle to the emitted radiation. The method of the cut-off function, and even more so that of Heitler, provides a kinematic correction rather than a dynamical solution of the problem." *A. J. Coleman.*

Bleuler, K., and Heitler, W. The reversal of time and the quantization of the longitudinal field in quantum electrodynamics. Progress Theoret. Physics **5**, 600–605 (1950).

The authors consider the theory of the quantized electromagnetic field interacting with a classical charge and current distribution. In this theory there exists a transformation of the variables, defined by a unitary operator S, which corresponds to a reversal of the sign of the time-coordinate. The form of S is simple so long as only transverse fields are considered, longitudinal fields being replaced by the equivalent Coulomb interaction between the charges. In the general case when both transverse and longitudinal field operators are transformed, the operator S is determined and is found to be no longer simple. But there exists an alternative method of quantizing the longitudinal field [S. N. Gupta, Proc. Phys. Soc. Sect. A. **63**, 681–691 (1950); these Rev. **12**, 67], using an indefinite metric in Hilbert space, which has the advantage of making S simple in all cases.

F. J. Dyson (Ithaca, N. Y.).

Ward, J. C. On the renormalization of quantum electrodynamics. Proc. Phys. Soc. Sect. A. **64**, 54–56 (1951).

The proof of the finiteness of the S-matrix in quantum electrodynamics given by the reviewer [Physical Rev. (2) **75**, 1736–1755 (1949); these Rev. **11**, 145] involved some extremely complicated and detailed algebraic manipulations. The author shows that the worst of the complications can be avoided by a simple argument, based on an identity which he had published earlier [ibid. **78**, 182 (1950); these Rev. **11**, 632]. *F. J. Dyson* (Ithaca, N. Y.).

Valatin, Jean G. Sur la seconde quantification. J. Phys. Radium (8) **12**, 131–141 (1951).

This is a full account of the material contained in an earlier note of the author [C. R. Acad. Sci. Paris **230**, 722–724 (1950); these Rev. **11**, 762] dealing with the application of Grassmann algebra to the second quantization for fermions. *I. E. Segal* (Chicago, Ill.).

Gamba, A. Proprietà di trasformazione dei campi di spin ½. Nuovo Cimento (9) **7**, 919–924 (1950).

A suggestion of Yang and Tiomno [Physical Rev. (2) **79**, 495–498 (1950); these Rev. **12**, 227] is formulated in a more general way. It is pointed out that not only four, but 16 different types of fields of spin ½ are possible if general inversion of space-time coordinates is admitted. The consequences for Fermi-type interactions are discussed.

E. Gora (Providence, R. I.).

Potier, Robert. Sur la dérivation variationnelle des équations générales décrivant les corpuscules de spin quelconque à masses multiples. C. R. Acad. Sci. Paris **232**, 1538–1540 (1951).

Potier, Robert. Sur les équations d'ondes des corpuscules à masses multiples. C. R. Acad. Sci. Paris **232**, 1647–1649 (1951).

Potier, Robert. Sur le calcul du quadrivecteur-courant dans la théorie des corpuscules de spin quelconque à masses multiples. C. R. Acad. Sci. Paris **232**, 1736–1738 (1951).

Matthews, P. T. Spinless mesons in the electromagnetic field. Physical Rev. (2) **80**, 292 (1950).

Formal rules are stated for writing down the matrix elements of the S-matrix in the quantized field theory of spin-zero mesons interacting with the electromagnetic field. It is then shown how the matrix elements can be made finite by a renormalization of mass and charge, provided that a term $\delta\lambda\phi^2\phi^{*2}$ is added to the Lagrangian density of the system, where ϕ, ϕ^* are the meson field and its Hermitian conjugate, and $\delta\lambda$ is a suitably chosen logarithmically divergent constant. The finiteness of the S-matrix is here verified only up to terms of order e^4, where e is the meson charge.

F. J. Dyson (Ithaca, N. Y.).

Matthews, P. T. Spinless mesons and nucleons in the electromagnetic field. Physical Rev. (2) **80**, 292–293 (1950).

The considerations of a previous letter [see the preceding review] are here extended to the case of charged spin-zero mesons and charged nucleons interacting simultaneously with each other and with the electromagnetic field. The mesons may be either scalar or pseudoscalar. It is found that the S-matrix is again finite up to terms of fourth order in the coupling constants, provided that the nucleon mass and the nucleon-meson coupling constant are renormalized together with the other constants. It is noteworthy that the single additional term $\delta\lambda\phi^2\phi^{*2}$ in the Lagrangian compensates the divergent meson-meson interactions arising in two quite distinct ways from the electromagnetic and nucleonic properties of the meson. *F. J. Dyson* (Ithaca, N. Y.).

Matthews, P. T. Renormalization of the meson-photon-nucleon interaction. Philos. Mag. (7) **42**, 221–227 (1951).

This is a detailed account of the work already reported briefly in the letter reviewed above. No new results are stated. *F. J. Dyson* (Ithaca, N. Y.).

Kinoshita, Toichiro, and Nambu, Yoichiro. On the interaction of mesons with the electromagnetic field. II. Progress Theoret. Physics **5**, 749–768 (1950).

This paper continues previous work [Kinoshita, same vol., 473–488 (1950); these Rev. **12**, 380] on second order corrections to the current operator for scalar and vector particles interacting with the electromagnetic field. In the previous paper the vacuum polarization term was evaluated. Here the work is completed by evaluating the vertex and self energy terms. The calculation goes through much as for the electron field with but few additional complications. For the scalar particle the divergences are again concealed in unobservable mass and charge renormalization. For the vector particle terms which are in principle measurable (including an anomalous quadrupole moment) are divergent.

K. M. Case (Ann Arbor, Mich.).

Rüdenberg, Klaus. Zur Theorie der starken Kopplung zwischen Nucleonen und pseudovektoriellen Mesonen. Helvetica Phys. Acta **24**, 89–126 (1951).

All other types of meson theory having failed to explain the observed nuclear forces, this paper initiates the study of the one hitherto unexplored case of a pseudovector meson field in strong coupling with the nucleon. The methods of Wentzel's paper on the vector meson are followed closely [Helvetica Phys. Acta. **16**, 551–596 (1943); these Rev. **5**, 278]. The nucleon together with part of the meson field forms the "complex nucleon" capable of isobaric states of higher spin and charge, whose energies are found to be $E_j=(2J)^{-1}j(j+1)$, where j is a half integer and $J=f^2/6\pi\mu a_0$ if $g=0$ and $J=g^2/3\pi\mu a_0$ if $f=0$, where f and g are the usual coupling constants, μ is the meson mass and a_0 is the nuclear "radius." Under the assumption that the interaction potential between proton and neutron is additive in the f and g parts (an assumption which is justified for weak but dubious for strong coupling) it is shown that in the first approximation the force between two nucleons in their lowest isobar state is $\frac{1}{4}$ the corresponding force in weak coupling theory and is therefore repulsive. It remains to be investigated whether this repulsive force is compensated by the effect of the higher isobar states which is known to be attractive.

A. J. Coleman (Toronto, Ont.).

Fukuda, Nobuyuki, and Miyazima, Tatuoki. The covariant theory of radiation damping. I. General formalism. Progress Theoret. Physics **5**, 849–860 (1950).

This paper describes an attempt to include the effects of radiation damping in meson theory (where they are expected to be large) within the covariant framework of Tomonaga and Schwinger. Writing the collision operator S as (1) $S=1-iR$ and expressing S in terms of Schwinger's reaction operator K [Physical Rev. (2) **74**, 1439–1461 (1948); these Rev. **10**, 345] by (2) $S=(1-\frac{1}{2}iK)/(1+\frac{1}{2}iK)$ gives the integral equation (3) $R=K-\frac{1}{2}iKR$. Using the known power series solution for S and the connection implied by (2) of S with K yields an approximate expression for K. This is inserted in (3). For the case of scattering of positive (negative) mesons on neutrons (protons) a rigorous solution of the integral equation is obtained. That damping effects do ensue is shown by tables giving results for typical values of the parameters. No discussion of the consistency of the procedure is included. *K. M. Case* (Ann Arbor, Mich.).

Abrikosov, A. A., and Halatnikov, I. M. Radiative corrections to Dirac's equation (in the nonrelativistic approximation). Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **21**, 69–78 (1951). (Russian)

The derivation of French and Weisskopff [Physical Rev. (2) **75**, 1240–1248 (1949)] of the radiation correction to the energy level of a bound electron is simplified by neglecting throughout the calculation those terms which do not involve the electromagnetic four-vector of the external field.

A. J. Coleman (Toronto, Ont.).

Marschall, Hans. Zur Theorie der anomalen Streuung von α-Teilchen an spinlosen Kernen. Z. Physik **128**, 635–656 (1950).

A treatment of alpha-particle scattering is proposed which does not presuppose any knowledge about the interior of the nucleus. The Coulomb potential is assumed to be valid up to a spherical nuclear surface of given radius. For the sake of simplicity only spinless nuclei are considered so that the total angular momentum of the system is determined by the orbital momentum of the incident particle. The alpha-particle will be able to penetrate the nucleus if a compound nucleus is formed for a particular angular momentum. Only one of the partial waves obtained in an expansion of the wave function in terms of spherical harmonics will then penetrate the nucleus. All the other partial waves must vanish on the surface of the nucleus; they are thus fully determined for given nuclear radius and charge. The Wentzel-Kramers-Brillouin approximation is used to represent both these waves and the partial wave which penetrates the nucleus, the "resonance-wave". In calculating the latter, two cases are distinguished: particle energies smaller or larger than height of the Coulomb potential on the nuclear surface. In the first case the anomalous scattering has predominantly resonance character. In the second case resonance scattering and direct scattering of the partial waves which do not penetrate the nucleus can lead to anomalies of the same order of magnitude. *E. Gora.*

Furry, W. H. On bound states and scattering in positron theory. Physical Rev. (2) **81**, 115–124 (1951).

The question of whether or not it is allowable to use bound state wave functions in positron theory is investigated, a possible difficulty arising because of the well-known infinities in the "nonrenormalized" version of this theory. By appropriate introduction of a new representation in which the field variables are expanded in terms of bound state eigenfunctions, it proves possible to treat the problem of polarization of the vacuum. With suitable approximations (Born approximation), the same result is obtained as one usually obtains using free particle wave functions. It is emphasized that the entire procedure is only valid for external fields which do not produce pairs, i.e., for those which leave the vacuum stable. Scattering is also considered in the bound state representation, and it is shown that it is possible to adopt different points of view in problems involving the coherent scattering of light by a bound electron. It is shown that one obtains the same numerical answer whether or not one applies the exclusion principle to the intermediate states.

J. M. Luttinger (Madison, Wis.).

Kolesnikov, N. N., and Blohincev, D. I. Bremsstrahlung of particles having an additional, spin-independent magnetic moment. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **20**, 979–986 (1950). (Russian)

Bremsstrahlung collision cross sections are calculated for scattering of particles possessing spin $\frac{1}{2}$ and an additional, spin-independent magnetic moment, by nuclei of given magnetic moment. The mass of the nuclei is assumed to be so large that their recoil can be neglected. In the relativistic domain the interaction between the magnetic moments leads to a rapid increase of the lowest order cross section, proportional to E_0^4, with the energy E_0 of the incident particle. Radiation damping is taken into account. Its influence becomes appreciable for $E_0\sim m_0c^2(\mu)^{-1}(\hbar c/e^2)^{\frac{1}{2}}$ where m_0c^2 is the rest energy of the particle, $(\hbar c/e^2)^{-1}\cong 1/137$ the fine structure constant, and μ the additional magnetic moment of the particle in Bohr's magnetons. For $\mu\sim 1$ radiation damping is thus likely to reduce the cross section considerably for $E_0\gtrsim m_0c^2$. If this result were still valid for high relativistic energies, the cross section would decrease there with E_0^{-4}.

E. Gora (Providence, R. I.).

Jost, R., Luttinger, J. M., and Slotnick, M. Distribution of recoil nucleus in pair production by photons. Physical Rev. (2) **80**, 189–196 (1950).

A covariant method of calculating cross sections is proposed which utilizes the unitarity of the S-matrix and the gauge invariance of the cross sections. This method is used to determine the angular and momentum distribution of the recoil nuclei in pair production by a photon. To evaluate the matrix elements the prescriptions of Feynman and Dyson are employed. The results are in disagreement with a recent experiment, particularly for small angles and high momentum transfers. A simplified and covariant calculation of the total pair production cross section is also given.

E. Gora (Providence, R. I.).

Irving, J. Non-physical solutions in classical finite electron theory. Proc. Phys. Soc. Sect. A. **63**, 1125–1131 (1950).

Using an approximate linear form of the equations of motion for an electron based on the Peierls-McManus theory [McManus, Proc. Roy. Soc. London Ser. A. **195**, 323–336 (1948); these Rev. **10**, 664], and taking a particular "influence" function satisfying all the conditions imposed by the theory, the author shows that nonphysical "runaway" solutions exist. This paper contains some discussion of the case of an arbitrary influence function. No function has been found for which the runaway solutions are absent.

N. Rosen (Chapel Hill, N. C.).

Kar, K. C., Sengupta, S., and Chatterji, P. P. The relativistic theory of scattering in Coulomb field by atoms. Indian J. Phys. **24**, 339–345 (1950).

The Mott-Sexl formula for relativistic electron scattering, including the term of order e^6 in the scattering cross-section, is derived by the wave-statistical method, in which the analogy with hydrodynamics is exploited.

H. C. Corben (Pittsburgh, Pa.).

Alfsen, Erik. Schéma lagrangien de la théorie de l'électron de Flint. C. R. Acad. Sci. Paris **232**, 699–701 (1951).

Gvozdover, S. D., and Magazanik, A. A. Study of paramagnetism of atomic nuclei by the method of magneto-spin resonance. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **20**, 705–721 (1 plate) (1950). (Russian)

The equations of motion for the components of the nuclear magnetic moment in nuclear magnetic resonance experiments have been given by F. Bloch [Physical Rev. (2) **70**, 460–474 (1946)]. He considers the case where a suitable radio frequency field is superposed upon a constant magnetic field, and takes into account relaxation effects. Heretofore his equations have been discussed for special cases only. The authors treat his equations of motion in as general a way as possible, and show that they lead to an integral equation of the Volterra type for the main component of the nuclear magnetic moment. To solve this integral equation the method of successive approximations is used. In principle the problem is thus solved, but even in simple special cases the calculations turn out to be so involved that only approximate formulae can be obtained. An interesting result is a dependence of the form of the induction signals upon the amplitude of the radio frequency field. In the second part of the paper the effects of a periodic modulation of the main magnetic field are considered. In the third part experimental results are quoted which confirm some of the theoretical conclusions. *E. Gora* (Providence, R. I.).

Tyablikov, S. V. The quantum theory of magnetic anisotropy. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **20**, 661–668 (1950). (Russian)

A semi-phenomenological theory of ferromagnetic anisotropy is obtained by introducing a symmetric tensor to describe the exchange interaction of electrons in incomplete shells. The simplest case, a uniaxial crystal, is taken. Perturbation methods are used to calculate the magnetization for three conditions: (1) field parallel to axis; (2) field perpendicular to axis, effect of field small compared to anisotropy of the tensor of exchange interaction; (3) strong field perpendicular to axis. The author asserts that the use of his method can be extended to give a complete treatment of the problem. *W. H. Furry* (Cambridge, Mass.).

Levin, M. L. On the theory of the Čerenkov effect. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **20**, 381–383 (1950). (Russian)

Two Čerenkov electrons are assumed to move along parallel paths in the z direction a distance ρ apart. The retarding force of one on the other is given by

$$f_z=-\frac{e^2}{c^2}\int_{\beta n>1}\left(1-\frac{1}{n^2\beta^2}\right)J_0\left(\sqrt{1-\frac{1}{n^2\beta^2}}\,\frac{\omega n}{c}\rho\right)\omega\,d\omega,$$

where the integral is over the frequencies satisfying the condition $\beta n>1$. *G. M. Volkoff* (Vancouver, B. C.).

Bardeen, J., and Shockley, W. Scattering of electrons in crystals in the presence of large electric fields. Physical Rev. (2) **80**, 69–71 (1950).

Da nach der Quantenmechanik der elektrische Widerstand von der Streuung der Elektronenwellen an dem von den Wärmeschwingungen gestörten Gitter verursacht wird, so entsteht die Frage, ob es gerechtfertigt ist den Streuprozess theoretisch einfach unter Vernachlässigung des den Strom verursachenden Feldes zu behandeln, wie man das bis jetzt getan hat. Zur Beantwortung dieser Frage gehen die Verfasser aus der Wellengleichung des Problems

$$[-(h^2/8\pi^2m)\Delta+V_0(\mathbf{r})-\mathbf{F}\cdot\mathbf{r}+H_I]\psi=i(h/2\pi)\partial\psi/\partial t$$

aus, wo $V_0(\mathbf{r})$ das periodische Kristallfeld, $\mathbf{F}$ die den Strom verursachende elektrische Feldintensität, und H_I den die Streuung verursachenden Wechselwirkungsoperator bedeutet. Das Problem ist also eine doppelte Störungsrechnung, die mit Hilfe der Variation der Konstanten gelöst wird, wobei als "ungestörte" Eigenfunktionen die Lösungen der Wellengleichung ohne das H_I enthaltende Glied benützt werden. Als Resultat folgt, dass bis zu einer Feldintensität von 6×10^5 volt/cm die Beeinflussung des Streuprozesses unwesentlich ist, also die bekannten Formeln für die Streuung benutzt werden können. *T. Neugebauer* (Budapest).

Vol'kenštein, F. F., and Bonč-Bruevič, V. L. On the behavior of electrons in ionic crystals. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **20**, 624–635 (1950). (Russian)

The theory of conducting electrons in a crystal lattice is generalized by taking into account exchange effects. The starting point is the Schroedinger equation for a many electron system, but only the interaction of two electrons is considered in deriving an explicit solution. It is shown that this solution contains not only electron states of the usual type which correspond to separate electrons, but also states which can be interpreted as indicating the appearance of electrons in "pairs." Such electron "pairs" might

adhere to neighboring ions and exert some influence upon secondary effects, for instance upon those due to the polarisation of the ion cores. *E. Gora* (Providence, R. I.).

Hauptman, H., and Karle, J. Relations among the crystal structure factors. Physical Rev. (2) **80**, 244–248 (1950).

Es wurden neue Beziehungen zwischen den Strukturfaktoren

$$(1) \qquad F_{\mathbf{h}} = V\int_V \rho(\mathbf{r}) \exp\left[-2\pi i\mathbf{h}\cdot\mathbf{r}\right]d\tau$$

und ihren Grössen, welche die Form von Ungleichungen annehmen, abgeleitet. Es wird der Grenzfall von punktförmigen Atomen behandelt, wo an Stelle von $F_{\mathbf{h}}$ die unitären Strukturfaktoren

$$(2) \qquad U_{\mathbf{h}} = \sum_{j=1}^{N} n_j \exp\left[-2\pi i\mathbf{h}\cdot\mathbf{r}_j\right]$$

bzw.

$$(3) \qquad U_{hkl} = \sum_{j=1}^{N} n_j \xi_j^h \eta_j^k \zeta_j^l = r_{hkl} e^{i\varphi_{hkl}}$$

mit $\xi_j = e^{-2\pi i x_j}$, $\eta_j = e^{-2\pi i y_j}$, $\zeta_j = e^{-2\pi i z_j}$, und $n_j = Z_j/\sum_{\mu=1}^{N} Z_\mu > 0$ als normalisierten Atomnummern, treten. Das Problem ist, Gleichungssystem (3) bei nur bekannten Grössen $|U_{hkl}| = |U_{\overline{hkl}}|$ nach den $x_j, \cdots$ aufzulösen [M. Avrami, Physical Rev. (2) **54**, 300–303 (1938)]. In diesem Grenzfall werden einige der Ungleichungen zu Gleichungen. Man wählt $6(N-1)$ Gleichungen von Typus (3), welche die $3N$ Unbekannten $\xi_j, \cdots$ und die $3(N-1)$ unbekannten Phasen φ_{hkl} enthalten. Das Problem ist überbestimmt und es erhebt sich die zusätzliche Aufgabe, die experimentell gefundenen Grössen so anzugleichen, dass das vollständige System (3) in sich widerspruchlos ist. Jede Atomkoordinate erscheint also als Wurzel einer algebraischen Gleichung, welche die Grössen der U_{hkl} und die Atomnummern n_j enthält. Die Verwendung Hermite'scher Formen liefert etwas allgemeinere Resultate als Avrami [loc. cit.] gefunden hat. Die Gleichungen sind sehr kompliziert. Bei der ganzen Untersuchung wird keine spezielle Symmetrie vorausgesetzt. Wegen der Komplexheit der erhaltenen Resultate bildet die allgemeine Anwendung ein noch zu lösendes Problem für sich. *W. Nowacki* (Bern).

Thermodynamics, Statistical Mechanics

Stanyukovič, K. P. On the increase of entropy in an infinite universe. Doklady Akad. Nauk SSSR (N.S.) **69**, 793–796 (1949). (Russian)

Set-theoretical arguments are given to support the view that the second law of thermodynamics is not applicable to an infinite universe provided it contains denumerable infinite types of "particles". The latter concept includes atoms, molecules, stars, etc. *L. Tisza* (Cambridge, Mass.).

Plotkin, I. R. On the increase of entropy in an infinite universe. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **20**, 1051–1053 (1950). (Russian)

The arguments of the paper reviewed above are strengthened by showing that no evolution towards equilibrium can be expected in an infinite universe even if one considers only one type of constituting particle. *L. Tisza.*

Schrödinger, E. Irreversibility. Proc. Roy. Irish Acad. Sect. A. **53**, 189–195 (1950).

The old philosophical difficulty of representing the macroscopically irreversible systems of thermodynamics in terms of reversible microscopic systems is considered. The type of resolution which is based on quantum mechanics, with its conception of the irreversibility of the measurement process, etc., does not appeal to the author (for reasons which do not appear to the reviewer to be cogent, the author not seeming to follow through consistently in the quantum terms of reference). He tries a more classical picture and introduces a mode of telling past from future by entropy changes—without, however, being concerned with other difficulties to which such a scheme might lead. The discussion is conversational rather than mathematical and hardly precise enough to make definitive appraisal easy.

B. O. Koopman (New York, N. Y.).

Brillouin, L. Maxwell's demon cannot operate: information and entropy. I. J. Appl. Physics **22**, 334–337 (1951).

It is shown that the fallacy in supposing that the second law of thermodynamics could be violated by Maxwell's demon resides in considering the wrong system—the gas only—rather than what is relevant to the question, the gas plus the radiation field which the demon must use to see the molecules. If the radiation field is that of a black body, the demon can see nothing. If it allows him to see the molecules, it is because some of it has directionality (as in the use of an electric torch) and hence has a lower entropy than in the black body case. But then in the process of seeing (and sorting) the molecules, the total entropy is increased: all that the demon accomplishes by his cleverness is to minimize the amount of increase. The entropy balance is studied quantitatively by methods similar to those used in illustrating the principle of indeterminacy. Attention is focused on the cycle: negative entropy→information (of demon)→negative entropy. Boltzmann's constant k is shown to represent the smallest possible amount of negative entropy required in an observation, either of the demon or of a scientist in his laboratory. *B. O. Koopman* (New York, N. Y.).

Brillouin, L. Physical entropy and information. II. J. Appl. Physics **22**, 338–343 (1951).

First, the author reviews briefly the Fermi-Dirac, Bose-Einstein, and classical, statistics, illustrating them by simple combinatorial problems of placing balls in boxes, and obtaining the Boltzmann type of expression for the entropy as a logarithm in the three cases, both exact and approximate. Next, he considers the information contained in a succession of signals, and by simple combinatorial reasoning, derives an expression for the amount of information as the negative of an entropy (negentropy) of the Fermi-Dirac type. Finally, the result obtained is compared with similar results of C. E. Shannon and his successors, and found to be in agreement with them. *B. O. Koopman.*

Verschaffelt, J. E. La thermomécanique des phénomènes de transport. J. Phys. Radium (8) **12**, 93–98 (1951).

The author gives a brief summary of the material contained in his numerous notes [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 17–38, 126–151, 325–343, 344–355, 500–517 (1948); **35**, 31–41, 293–310, 311–324, 559–567, 976–994 (1949); **36**, 290–301, 548–560 (1950)] on the phenomenological theory of energy transport. *C. Truesdell.*

Isihara, Akira. Transport phenomena in degenerate systems. I. General theory in a mixed system. J. Phys. Soc. Japan **5**, 213–217 (1950).

The statistical method of Enskog and Chapman, which was generalized by Uehling and Uhlenbeck to incorporate the transport phenomena into quantum statistics, is here adapted to the degenerate case in which the quantum effects are no longer small corrections. The resultant formulas for the gas coefficients have the same form as usual; only the significance of the parameters is changed. *F. London.*

Isihara, Akira. Transport phenomena in degenerate systems. II. Degenerate Bose-Einstein gas. J. Phys. Soc. Japan **5**, 217–222 (1950).

The method of the preceding paper is applied to calculate the transport coefficients in a degenerate Bose-Einstein gas. While for high temperatures the gas coefficients are independent of density this is no longer true for very low temperature. The results are interpreted to mean that only the "normal" component of the degenerating Bose-Einstein gas transfers momentum, whereas the mean free path depends on the density of the condensed phase also. The numerical results for the viscosity are compared with the measured viscosity of liquid helium and, of course, only qualitative agreement has been found. *F. London.*

Whalley, E., and Winter, E. R. S. The elementary theory of thermal diffusion. Trans. Faraday Soc. **46**, 517–526 (1950).

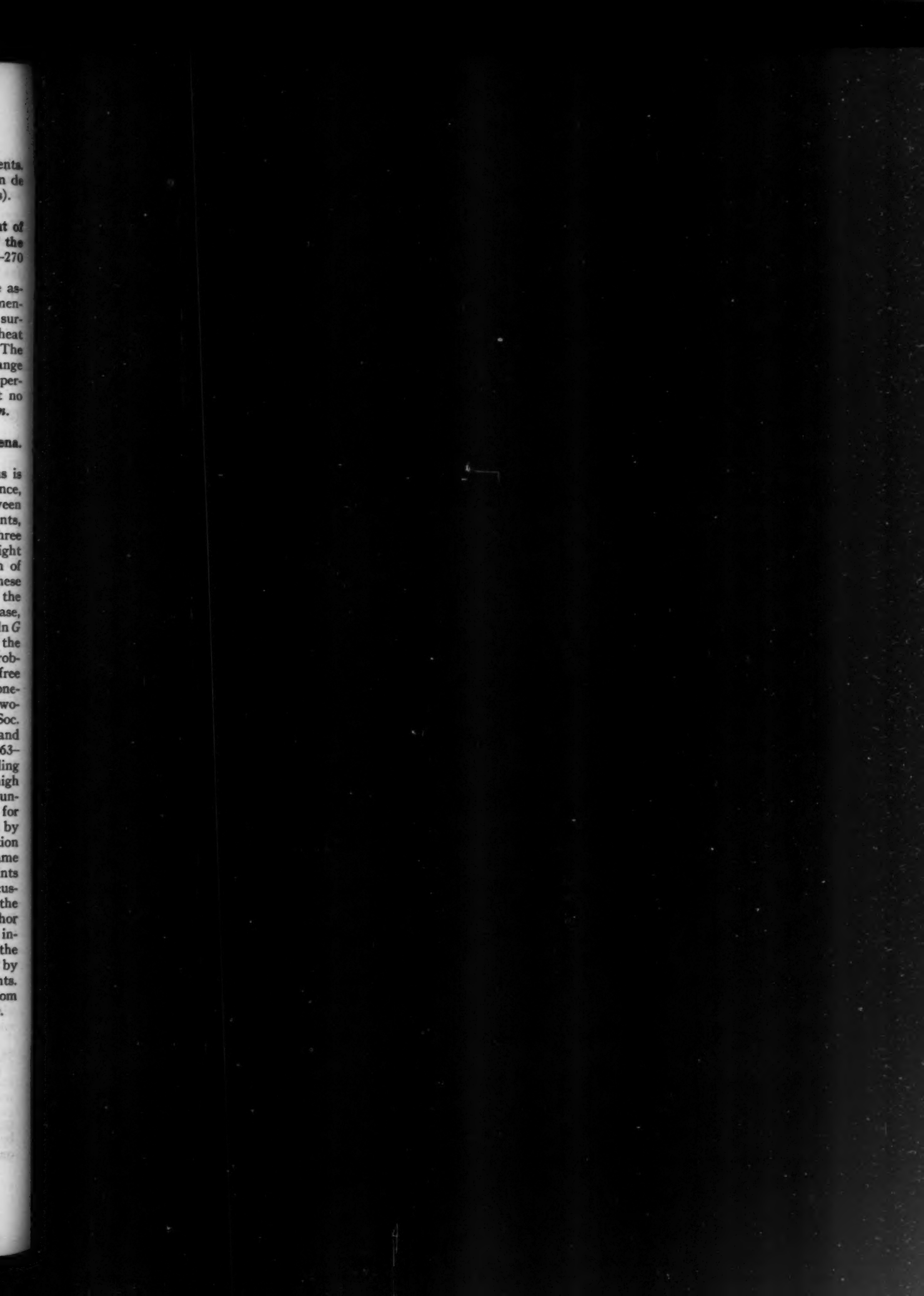
This is a simplified treatment of the theory of transport of multicomponent gas mixtures. It is assumed that the molecules are hard spheres. The emphasis is on the derivation of the thermal diffusion constants. In cases in which the masses and diameters of the molecules contribute in the same direction toward thermal diffusion, the results of the simplified theory are in good agreement with those of Chapman and Enskog. *E. W. Montroll* (College Park, Md.).

Heinrich, G., und Klemenc, A. Zur Behandlung der Diffusionsvorgänge in der Atmosphäre. Acta Physica Austriaca **4**, 160–169 (1950).

Dans un travail antérieur [Monatsh. Chem. **79**, 253–258 (1948)] les auteurs ont étudié le problème stationnaire de la distribution de la concentration pour un mélange gazeux à deux constituants dans le champ gravifique terrestre pour une distribution donnée de la température. Les résultats ont été appliqués à la distribution de l'argon dans l'atmosphère terrestre. Dans ce mémoire les auteurs généralisent le problème au cas non stationnaire avec un nombre quelconque des constituants. On obtient un système intégro-différentiel. Étant donné que l'on ne connait pas la grandeur des constantes de diffusion en fonction de la pression et de la température, les équations obtenues peuvent donner seulement une idée assez grossière sur l'ordre de grandeur des phénomènes. On trouve aisément que les concentrations dans les états stationnaire et non stationnaire sont différents. Comme application les auteurs étudient la distribution de l'hélium dans l'atmosphère. *M. Kiveliovitch* (Paris).

Ichimura, Hiroshi. A statistical mechanical treatment of conduction electrons. An attempt to the theory of the superconductivity. J. Phys. Soc. Japan **4**, 265–270 (1949).

The ordinary Fermi distribution is modified by the assumption of a cooperative interaction supposed in momentum space to act between the electrons near the Fermi surface. This assumption is shown to lead to a specific heat anomaly similar to the one found in superconductors. The author thinks that the model can explain the long range order of the mean momentum maintained in the superconducting state in presence of a magnetic field. But no attempt is made to prove this statement. *F. London.*

Kikuchi, Ryoichi. A theory of cooperative phenomena. Physical Rev. (2) **81**, 988–1003 (1951).

The Ising model for one, two, and three dimensions is studied by considering various configurations; for instance, in the one-dimensional case, a point and the bond between two adjacent points, in the two-dimensional case by points, bonds, squares, and "angles", the latter consisting of three points and two bonds associated with an elementary right angle of the lattice. Interrelated probabilities for each of these configurations are assumed and in terms of these probabilities the number G of ways of constructing the lattice is obtained exactly in the one-dimensional case, approximately in the remaining cases. The entropy $S=k \ln G$ and the energy E can therefore be expressed in terms of the probabilities of the basic configurations and these probabilities can be specified by the condition that the "free energy" $\mu=E-TS$ should be a minimum. In the one-dimensional case, this yields the precise result. In the two-dimensional case, the Bethe approximation [Proc. Roy. Soc. London. Ser. A. **150**, 552–575 (1935)] or the Kramers and Wannier approximation [same Rev. (2) **60**, 252–262, 263–276 (1941); these Rev. **3**, 63, 64] are obtained depending upon the choice of the configurations. The state of high order is characterized by the existence of two solutions, unsymmetric in the spin direction, of the conditions for minimizing μ, and the transition temperature is obtained by assuming these to run together into a symmetric solution associated with the disordered state. Based on the same configurations, i.e. squares, "angles", bonds, and points used in the two-dimensional case, the corresponding discussion for the three-dimensional case is carried through for the disordered state and the transition temperature. The author points out that in order to avoid the approximations involved in his procedure for computing the entropy in the two-dimensional case, one must build up the lattice by bordering rectangles by one-dimensional strips of points. Triangular and tetrahedral lattices are also considered from the same point of view. *F. J. Murray.*